



NPTEL ONLINE CERTIFICATION COURSES

Finite Element Method

**Week 10: Two-dimensional problems in linear elasticity
(Examples and computer implementation)**

Lecture 47: Torsion

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This Week

Example of 2D scalar field problem - Torsion

Higher order triangular elements, area coordinates

2D Elasticity problems – Example and computer implementation

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$$\nabla \cdot (k \nabla u) = -f \text{ in } \Omega$$

Governing equations

$$u = \hat{u} \text{ on } \Gamma_u$$

Essential BC

$$k \partial u / \partial n + \beta(u - u_\infty) = q \text{ on } \Gamma_q$$

Natural BC

Field of applications	Primary variable u	Material constant k	Source variable f	Secondary variable $q, \partial u / \partial x, \partial u / \partial y$
Heat Transfer	Temperature T	Conductivity k	Heat source g	Heat flow due to conduction $k \partial T / \partial n$ Convection $h(T - T_\infty)$
Irrotational flow of an ideal fluid	Stream function Ψ Velocity potential Φ	Density (ρ)	Mass production σ	Velocities $\partial \Psi / \partial x = -v \quad \partial \Psi / \partial y = u$ $\partial \Phi / \partial x = u \quad \partial \Phi / \partial y = v$
Ground water flow	Piezometric head Φ	Permeability K	Recharge f (pumping, $-f$)	Seepage $q = k \partial \Phi / \partial n$ Velocities $k \partial \Phi / \partial x = -u \quad k \partial \Phi / \partial y = -v$
Torsion of cylindrical members with constant cross-section	Stress function Ψ	$k = 1$ $G = \text{shear modulus}$	$f = 2$ $\theta = \text{angle of twist per unit length}$	$G \theta \partial \Psi / \partial x = -\sigma_{yz}$ $G \theta \partial \Psi / \partial y = \sigma_{xz}$
Electrostatics	Scalar potential Φ	Dielectric constant (ϵ)	Charge density ρ	Displacement flux D_n
Magnetostatics	Magnetic potential Φ	Permeability (μ)	Charge density ρ	Displacement flux B_n
Transverse deflection of elastic membrane	Transverse deflection u	Tension in member T	Transversely distributed load	Normal force q

Recall: Torsion of Circular Shaft

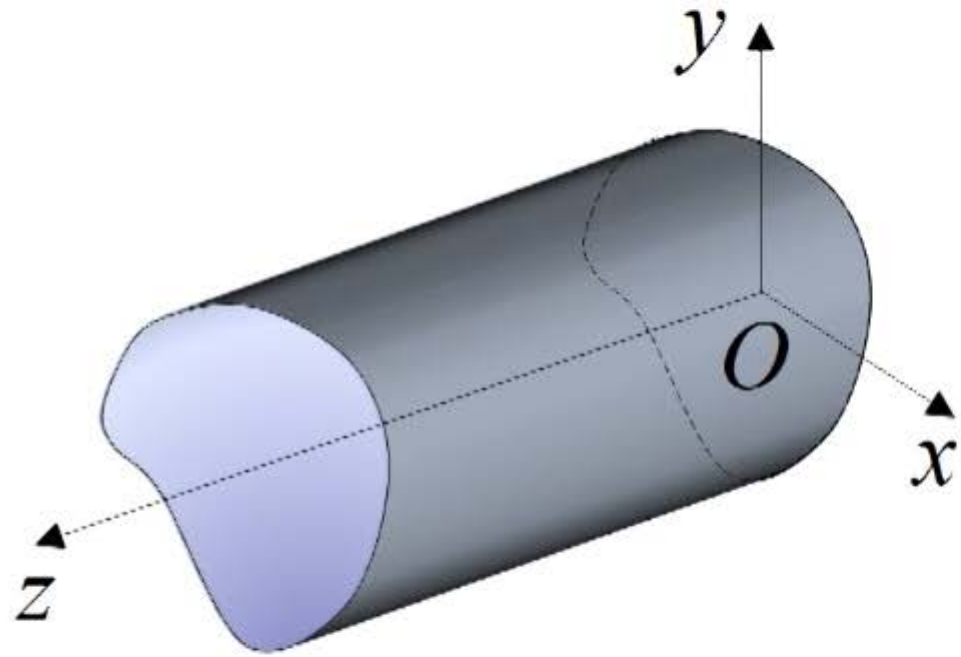


Each cross section rotates as a rigid body about the longitudinal axis.

Rotation is a linear function of axial coordinate.

Plane sections perpendicular to the longitudinal axis before deformation remain plane (and perpendicular to the longitudinal axis) after deformation.

Recall: Torsion of Non-Circular Shaft

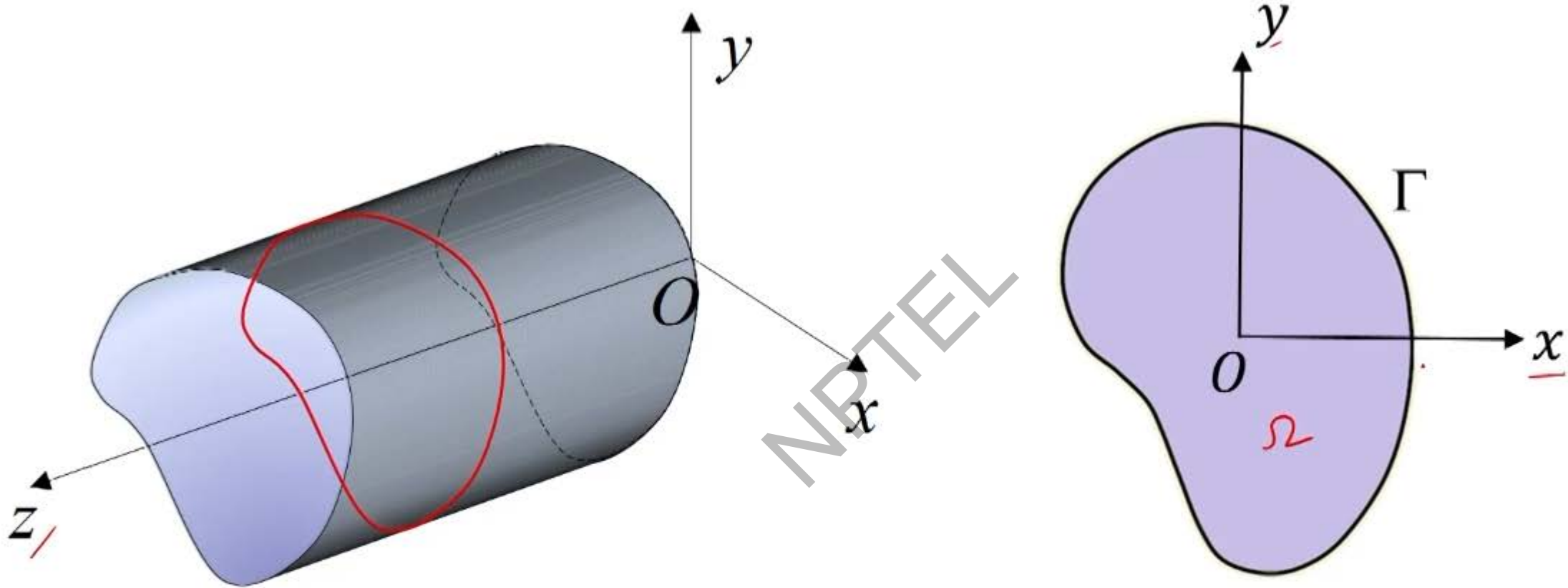


The projection of each section the x - y plane rotates as a rigid body about the longitudinal axis.

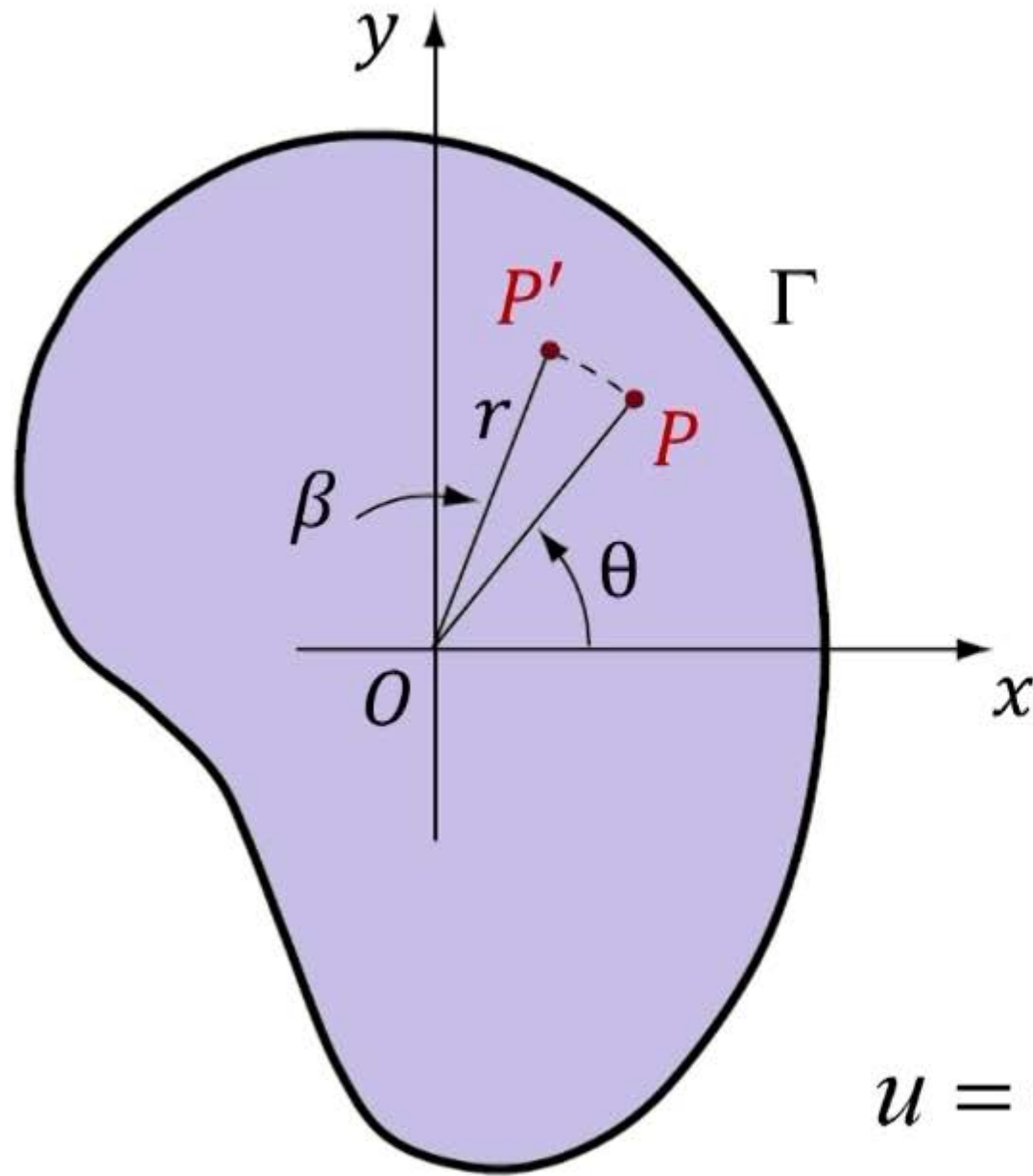
Rotation of the projected section is a linear function of axial coordinate.

Plane sections may not remain plane after deformation – warping

Torsion: Displacement Components



Torsion: Displacement Components



$$PP' = r\beta$$

$$u = -r\beta \sin \theta = -\beta y$$

$$v = r\beta \cos \theta = \beta x$$

$$\beta = \theta z$$

$$u = -\theta zy \quad v = \theta zx \quad w = \theta \varphi(x, y)$$

Recall: Stress, Strains, Elasticity etc.

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\varepsilon_{xy} = \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$

$$\varepsilon_{yz} = \frac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)$$

$$\varepsilon_{zx} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)$$

$$u = -\theta \, zy$$

$$v = \theta \, zx$$

$$w = \theta \, \varphi(x, y)$$

Recall: Stress, Strains, Elasticity etc.

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = 0$$

$$\varepsilon_{yz} = \frac{\theta}{2} \left(\frac{\partial \varphi}{\partial y} + x \right)$$

$$\varepsilon_{zx} = \frac{\theta}{2} \left(\frac{\partial \varphi}{\partial x} - y \right)$$

$$u = -\theta zy$$

$$v = \theta zx$$

$$w = \theta \varphi(x, y) .$$

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Torsion: Stress Formulation (Governing Equation)

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = 0$$

$$\varepsilon_{yz} = \frac{\theta}{2} \left(\frac{\partial \varphi}{\partial y} + x \right)$$

$$\varepsilon_{zx} = \frac{\theta}{2} \left(\frac{\partial \varphi}{\partial x} - y \right)$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0$$

$$\sigma_{yz} = G\theta \left(\frac{\partial \varphi}{\partial y} + x \right)$$

$$\sigma_{zx} = G\theta \left(\frac{\partial \varphi}{\partial x} - y \right)$$

Torsion: Stress Formulation (Governing Equation)

Recall: $\frac{\partial \sigma_{ji}}{\partial x_j} + b_i = 0$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0$$

$$\sigma_{yz} = G\theta \left(\frac{\partial \varphi}{\partial y} + x \right)$$

$$\sigma_{zx} = G\theta \left(\frac{\partial \varphi}{\partial x} - y \right)$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + b_x = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + b_y = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0$$

Torsion: Stress Formulation (Governing Equation)

Recall: $\frac{\partial \sigma_{ji}}{\partial x_j} + b_i = 0$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0$$

$$\sigma_{yz} = G\theta \left(\frac{\partial \varphi}{\partial y} + x \right)$$

$$\sigma_{zx} = G\theta \left(\frac{\partial \varphi}{\partial x} - y \right)$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0$$

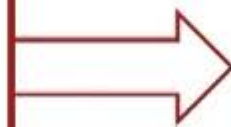
Torsion: Stress Formulation (Governing Equation)

Define stress function as

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} - x$$

$$\frac{\partial}{\partial x} \left(G \theta \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(G \theta \frac{\partial \phi}{\partial y} \right) = 0$$

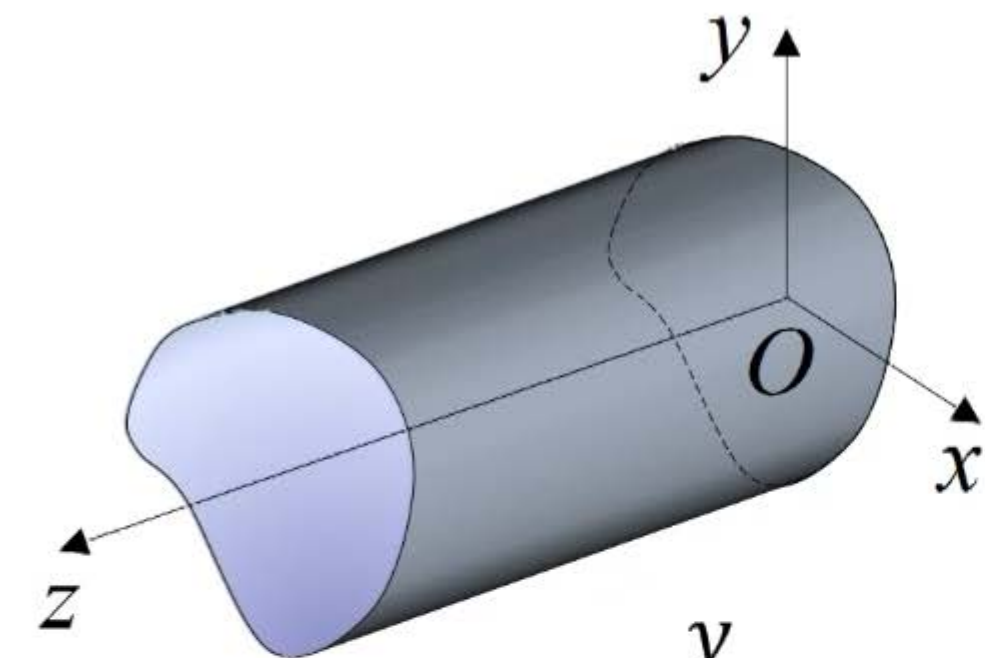
$$\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} - y$$



$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -2$$

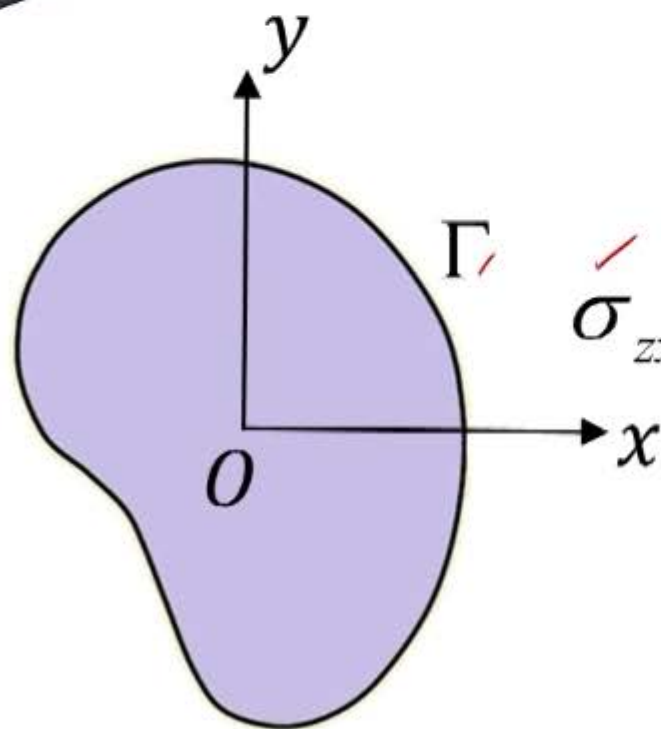
Recall: Poisson's equation $\nabla \cdot (k \nabla T) = -f$

Torsion: Stress Formulation (Governing Equation)



$$\sigma_{yz} = G\theta \left(\frac{\partial \varphi}{\partial y} + x \right) \quad \sigma_{zx} = G\theta \left(\frac{\partial \varphi}{\partial x} - y \right)$$

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \varphi}{\partial y} - x \quad \frac{\partial \psi}{\partial y} = \frac{\partial \varphi}{\partial x} - y$$



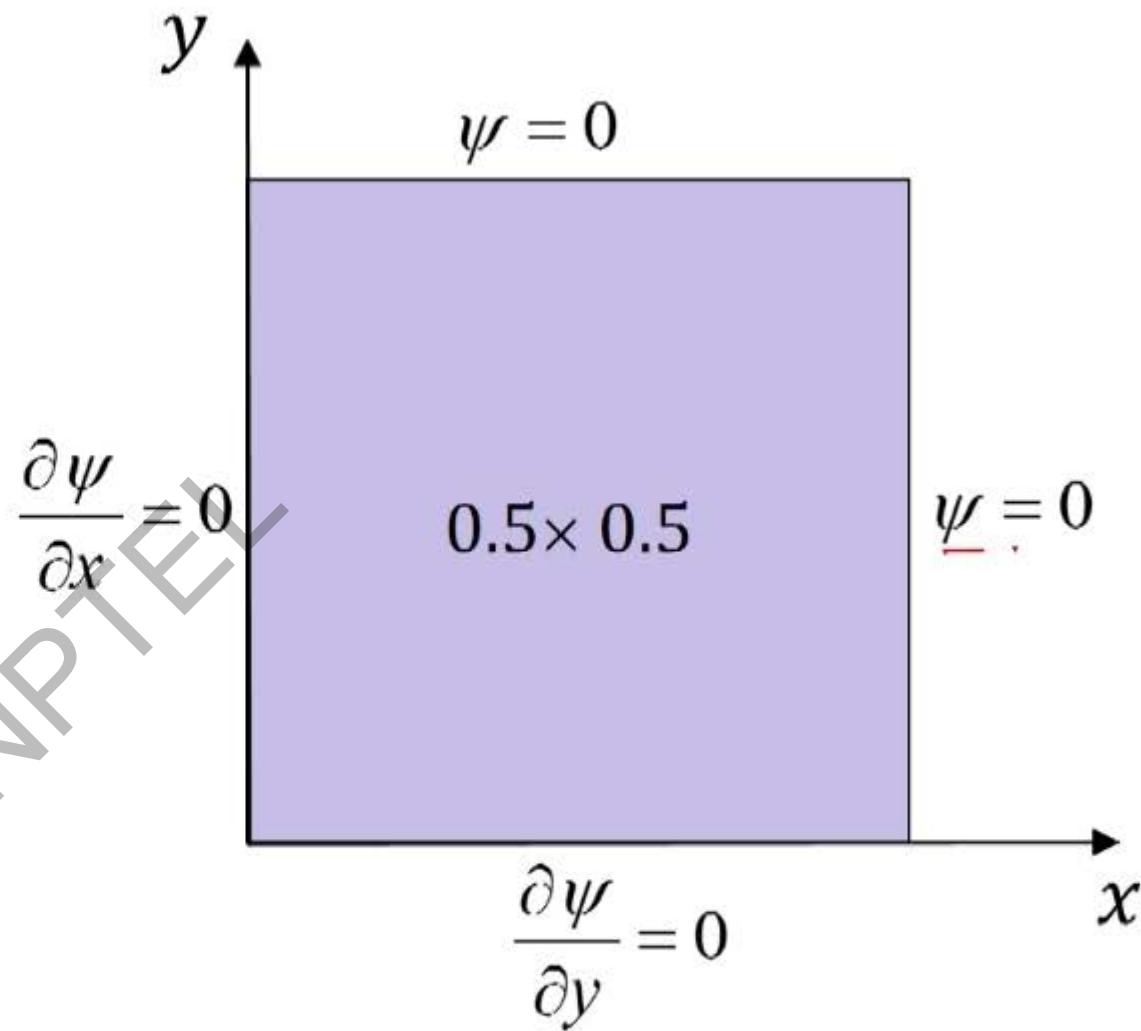
$$\sigma_{zx} n_x + \sigma_{yz} n_y = 0 \quad \Rightarrow \quad \frac{\partial \psi}{\partial y} n_x - \frac{\partial \psi}{\partial x} n_y = 0$$

$$\Rightarrow \psi = \text{Constant on } \underline{\Gamma}$$

Torsion: Boundary value problem

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -2 \quad \text{in } \Omega$$

$$\psi = 0 \quad \text{on } \Gamma$$





NPTEL ONLINE CERTIFICATION COURSES

Finite Element Method

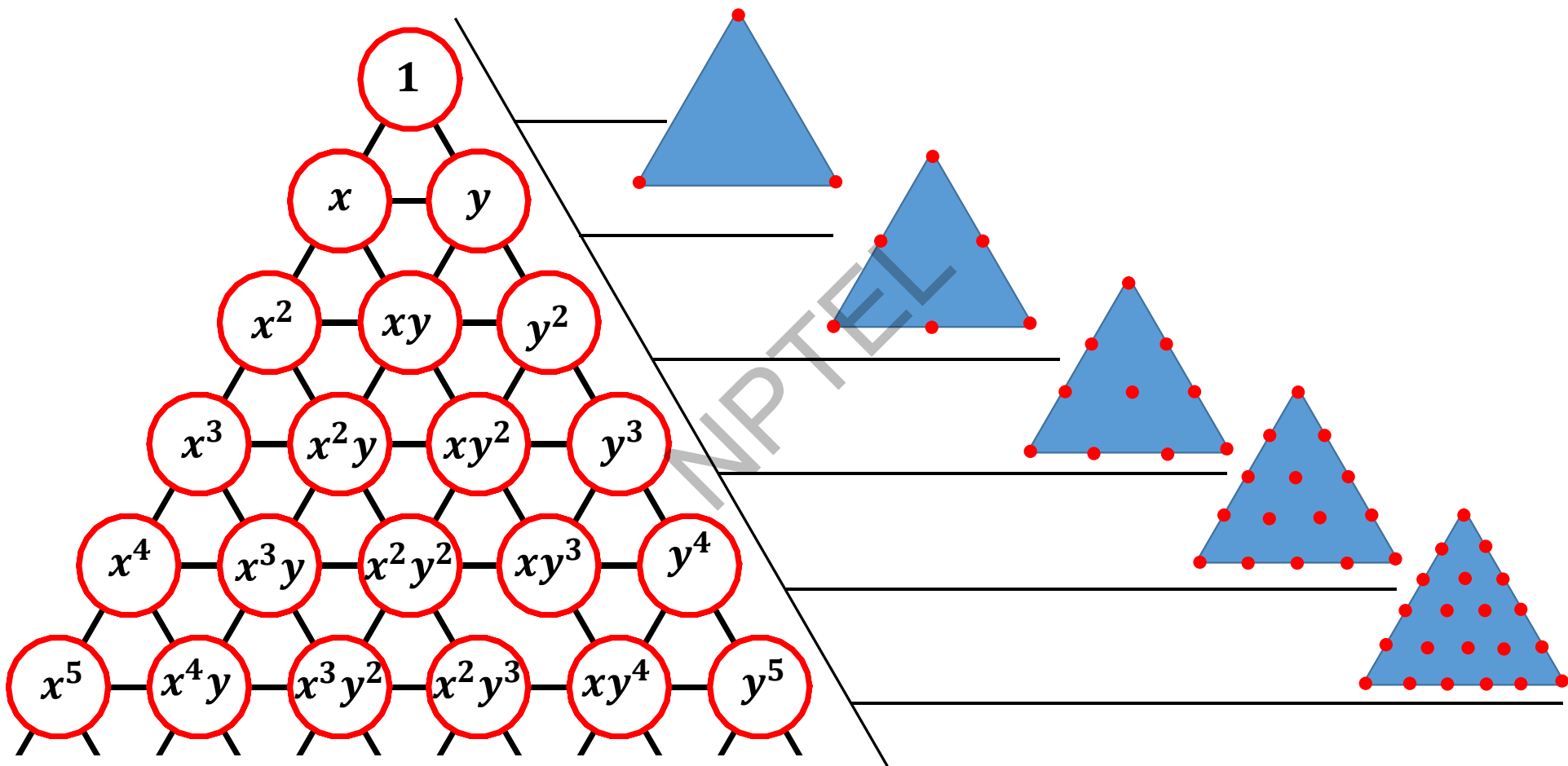
Week 10: Two-dimensional problems in linear elasticity

Lecture 48: Triangular Elements

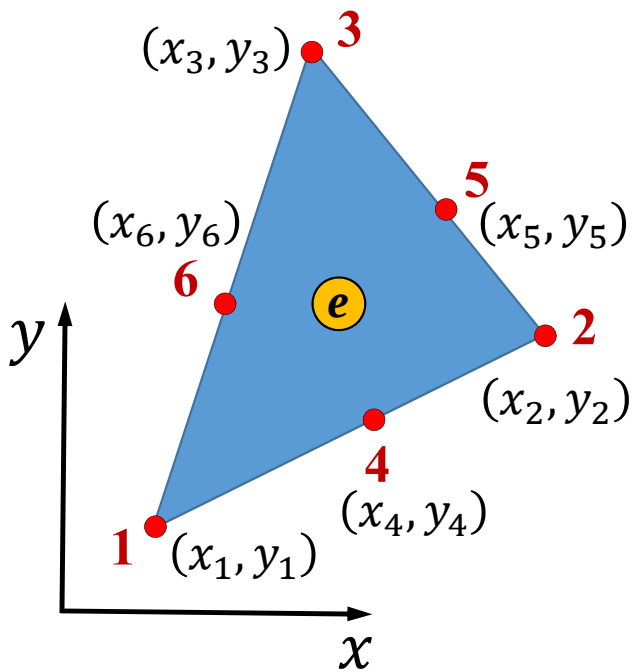
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Recall: Lagrange Simplexes



Shape Function For 6-Node Triangle



Polynomial approximation

$$u^e(x, y) = c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2$$

Interpolation

$$u^e(x_1, y_1) = c_1 + c_2x_1 + c_3y_1 + c_4x_1^2 + c_5x_1y_1 + c_6y_1^2$$

$$u^e(x_2, y_2) = c_1 + c_2x_2 + c_3y_2 + c_4x_2^2 + c_5x_2y_2 + c_6y_2^2$$

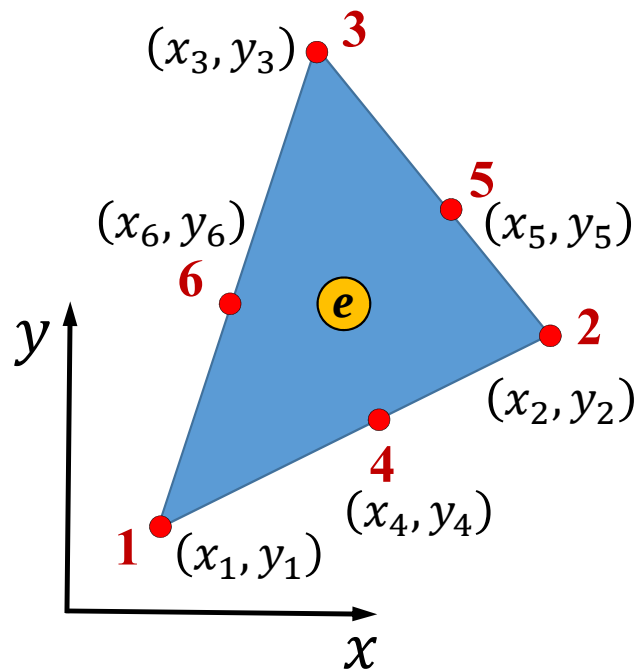
$$u^e(x_3, y_3) = c_1 + c_2x_3 + c_3y_3 + c_4x_3^2 + c_5x_3y_3 + c_6y_3^2$$

$$u^e(x_4, y_4) = c_1 + c_2x_4 + c_3y_4 + c_4x_4^2 + c_5x_4y_4 + c_6y_4^2$$

$$u^e(x_5, y_5) = c_1 + c_2x_5 + c_3y_5 + c_4x_5^2 + c_5x_5y_5 + c_6y_5^2$$

$$u^e(x_6, y_6) = c_1 + c_2x_6 + c_3y_6 + c_4x_6^2 + c_5x_6y_6 + c_6y_6^2$$

Shape Function For 6-Node Triangle



Polynomial approximation

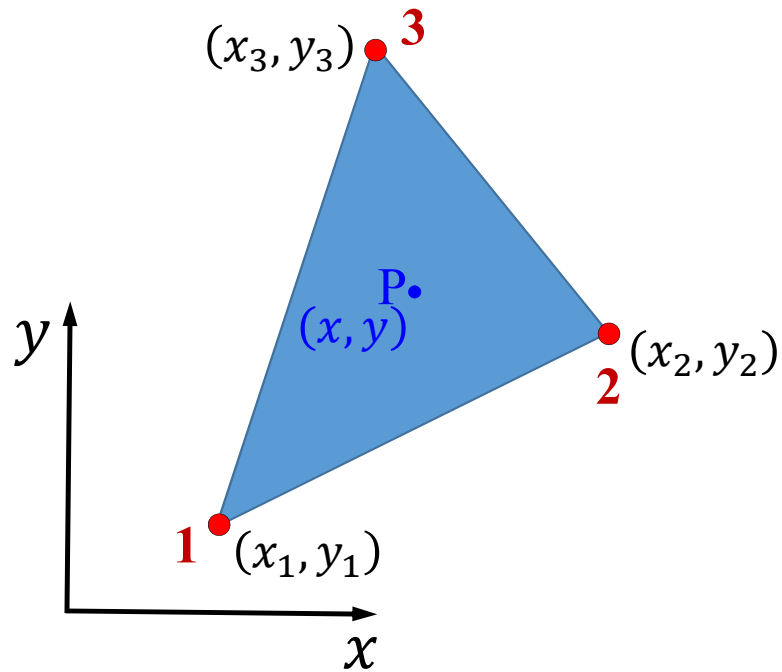
$$u^e(x, y) = c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2$$

Interpolation

$$\begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 \\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 \\ 1 & x_3 & y_3 & x_3^2 & x_3y_3 & y_3^2 \\ 1 & x_4 & y_4 & x_4^2 & x_4y_4 & y_4^2 \\ 1 & x_5 & y_5 & x_5^2 & x_5y_5 & y_5^2 \\ 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix} = \begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \\ u_5^e \\ u_6^e \end{Bmatrix}$$

Algebraically very tedious

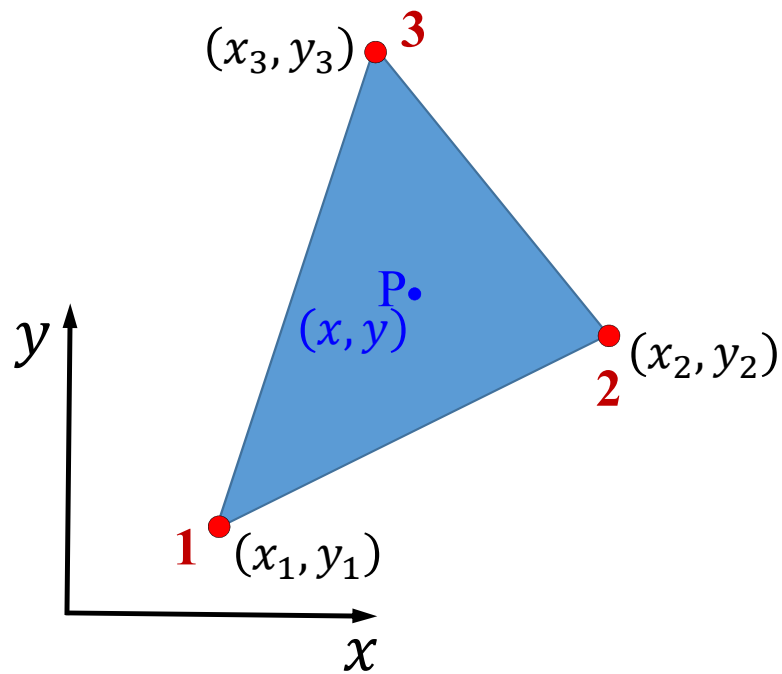
Recall: Shape Function For 3-Node Triangle



$$\phi_i(x, y) = \frac{1}{2A} (\alpha_i + \beta_i x + \gamma_i y)$$

$$\left. \begin{aligned} \alpha_i &= x_j y_k - x_k y_j \\ \beta_i &= y_j - y_k \\ \gamma_i &= -(x_j - x_k) \end{aligned} \right\} \begin{array}{l} i \neq j \neq k \\ i, j \text{ and } k \text{ permute in} \\ \text{a natural order} \end{array}$$

Recall: Shape Function For 3-Node Triangle



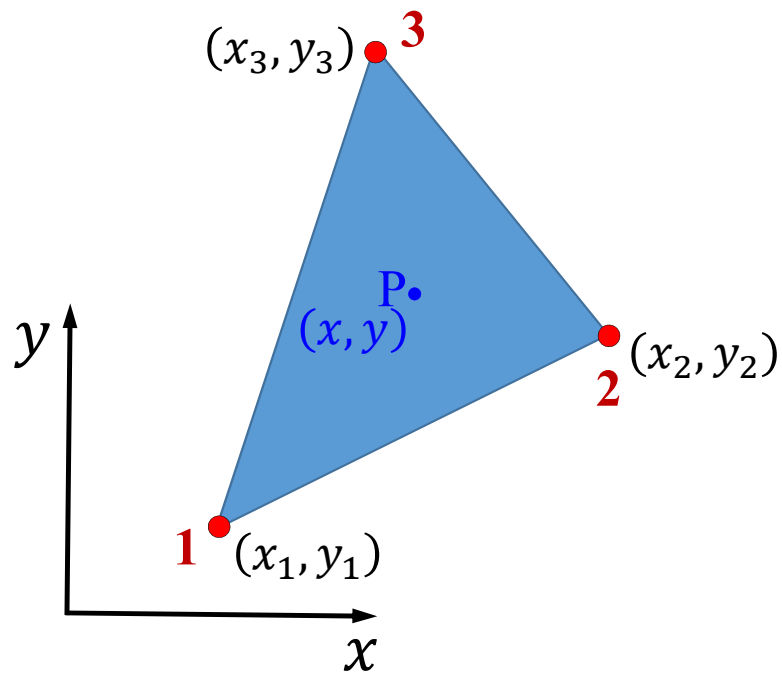
$$\phi_1(x, y) = \frac{1}{2A} (\alpha_1 + \beta_1 x + \gamma_1 y)$$

$$\left. \begin{aligned} \alpha_1 &= x_2 y_3 - x_3 y_2 \\ \beta_1 &= y_2 - y_3 \\ \gamma_1 &= -(x_2 - x_3) \end{aligned} \right\}$$



$$\phi_1(x, y) = \frac{1}{2A_e} [(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x - (x_2 - x_3)y]$$

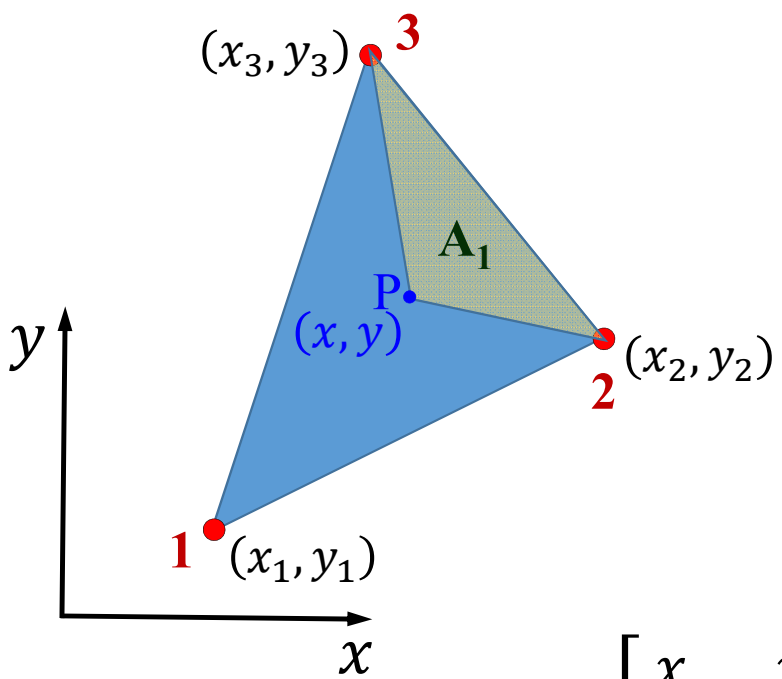
Recall: Shape Function For 3-Node Triangle



$$\phi_1(x, y) = \frac{1}{2A} \begin{bmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

$$\phi_1(x, y) = \frac{1}{2A_e} [(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x - (x_2 - x_3)y]$$

Recall: Shape Function For 3-Node Triangle

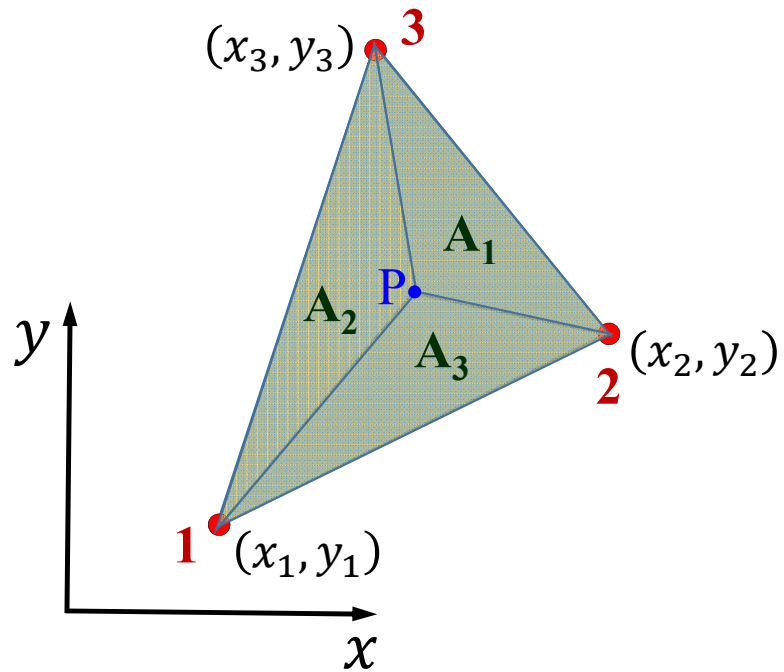


$$\phi_1(x, y) = \frac{1}{2A} \begin{bmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

$$A_1 = \frac{1}{2} \begin{bmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

$$\Rightarrow \phi_1(x, y) = \frac{\frac{1}{2} \begin{bmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}}{A} \Rightarrow \phi_1(x, y) = \frac{A_1}{A}$$

Recall: Shape Function For 3-Node Triangle



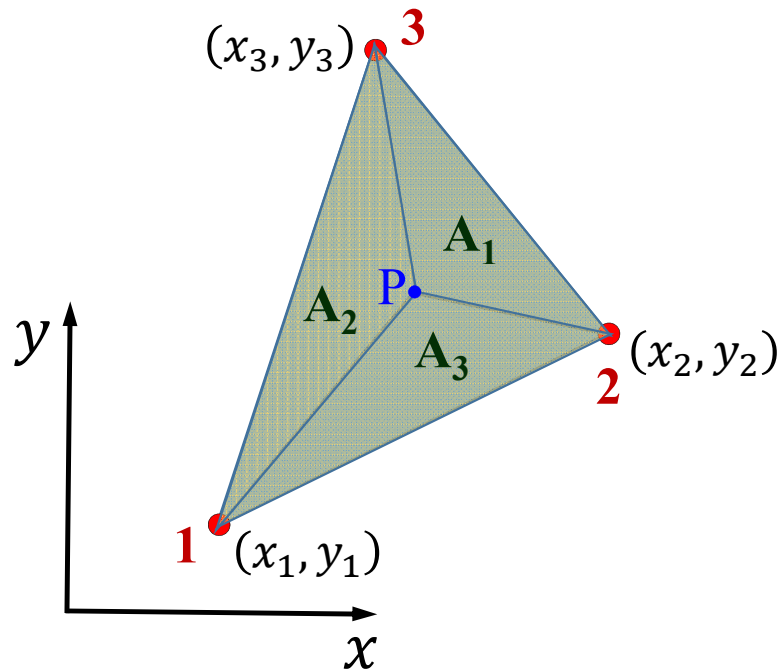
$$\phi_1 = \frac{A_1}{A}$$

Similarly

$$\phi_2 = \frac{A_2}{A}$$

$$\phi_3 = \frac{A_3}{A}$$

Recall: Shape Function For 3-Node Triangle



$$L_1 = \frac{A_1}{A}$$

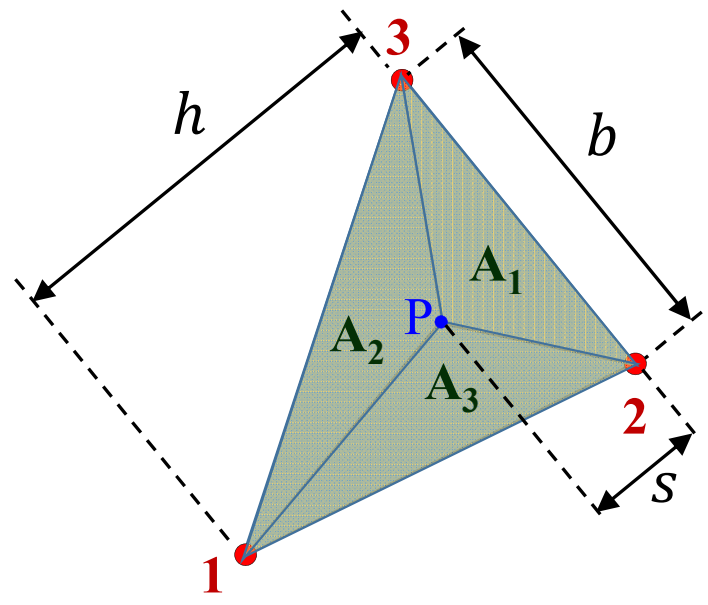
Similarly

$$L_2 = \frac{A_2}{A}$$

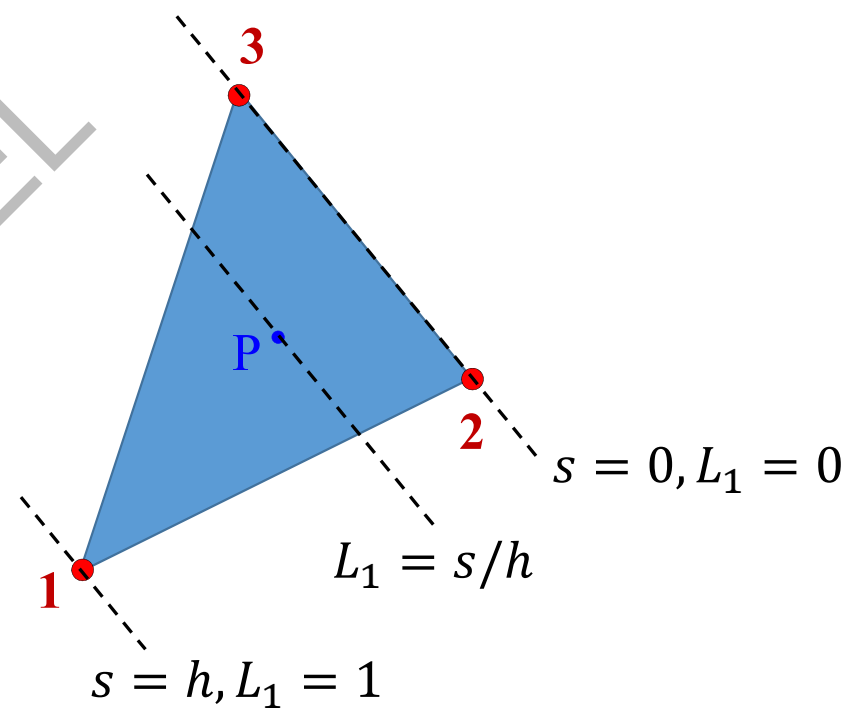
$$L_3 = \frac{A_3}{A}$$

L_1 , L_2 and L_3 are called area coordinates

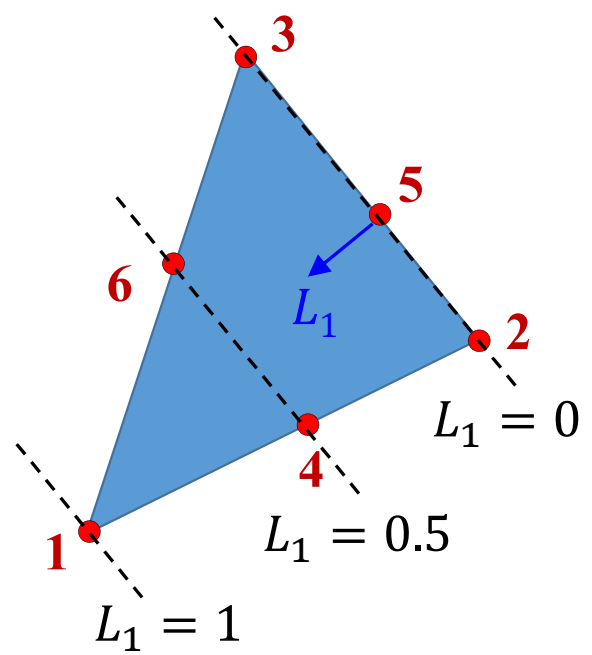
Area Coordinates



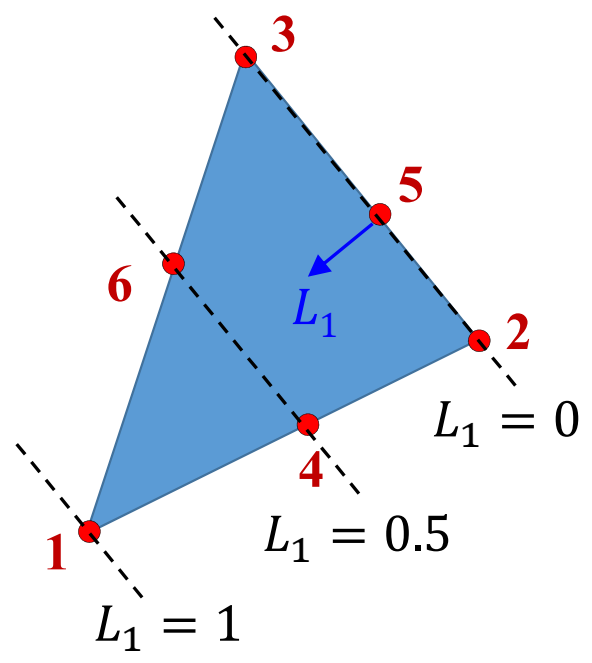
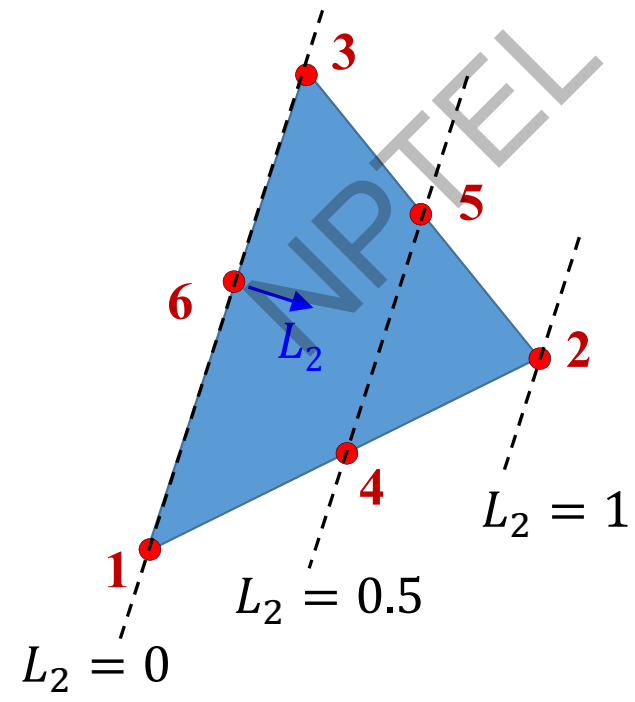
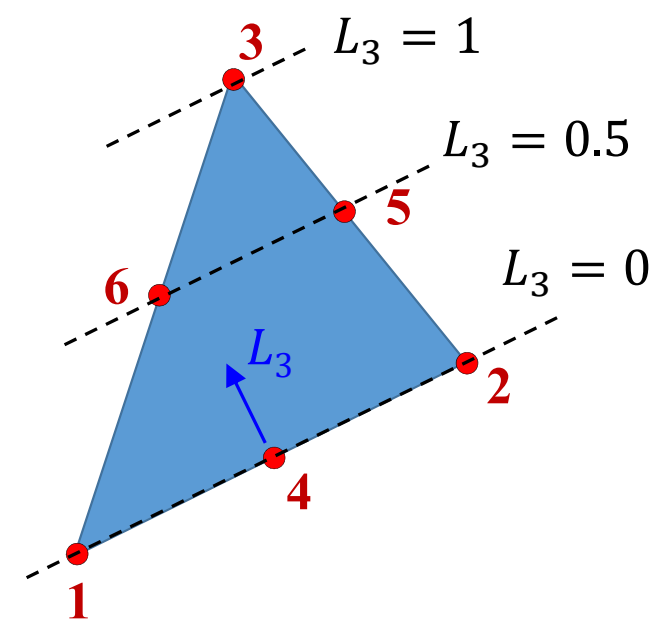
$$L_1 = \frac{A_1}{A} = \frac{bs/2}{bh/2} = \frac{s}{h}$$



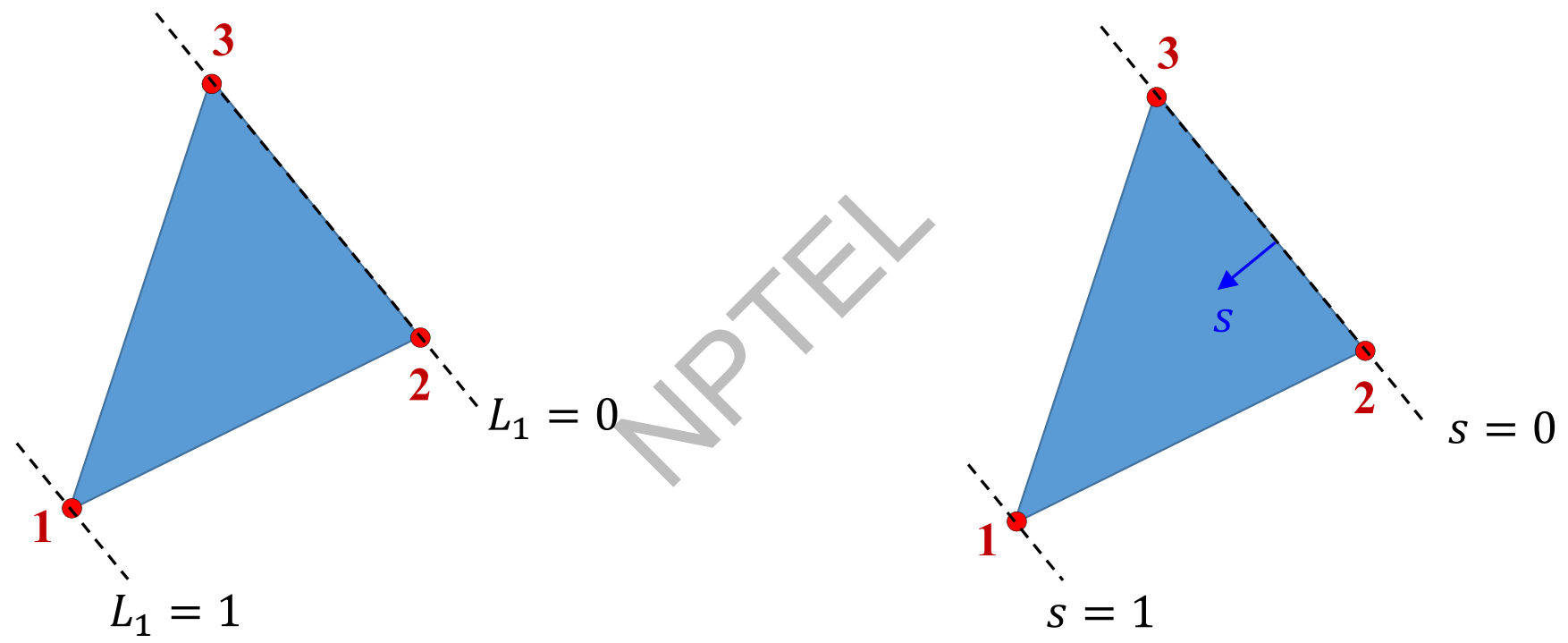
Area Coordinates for 6-Node Triangle



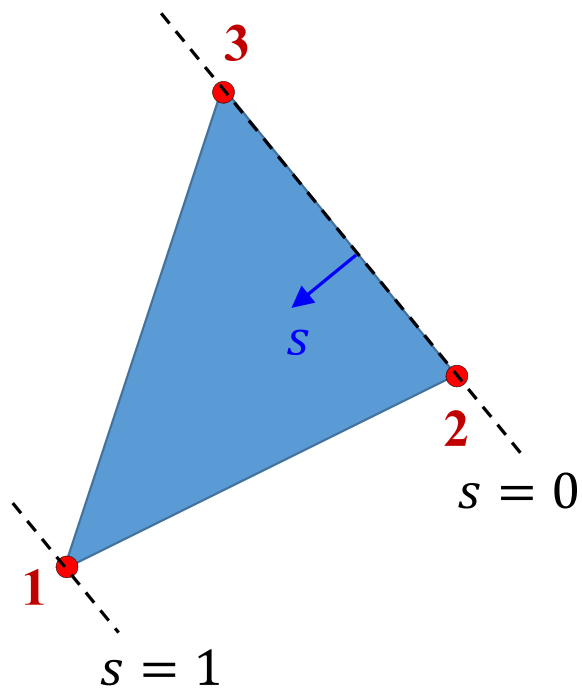
Area Coordinates



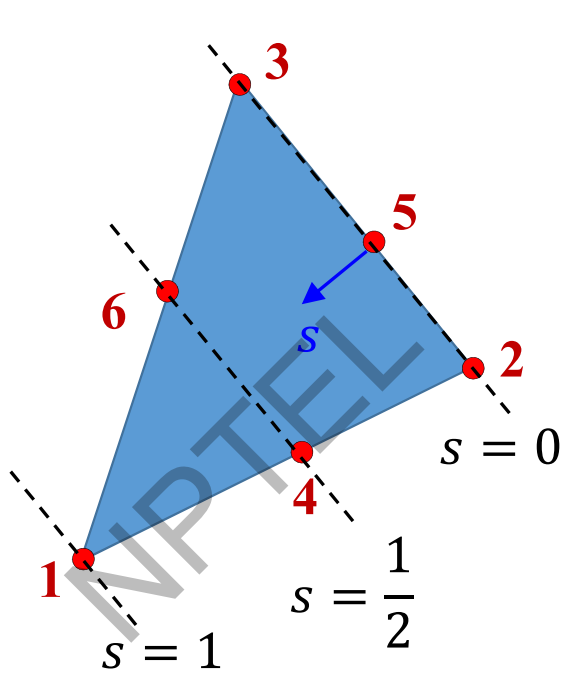
Area Coordinates



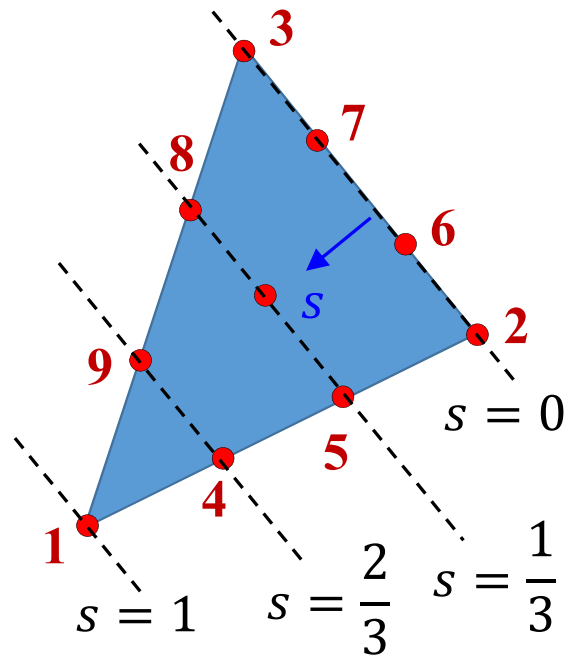
Area Coordinates



2 nodes per side
 (equally spaced) - linear
 Total number of nodes
 $2 + 1 = 3$



3 nodes per side
 (equally spaced) - quadratic
 Total number of nodes
 $= 3 + 2 + 1 = 6$

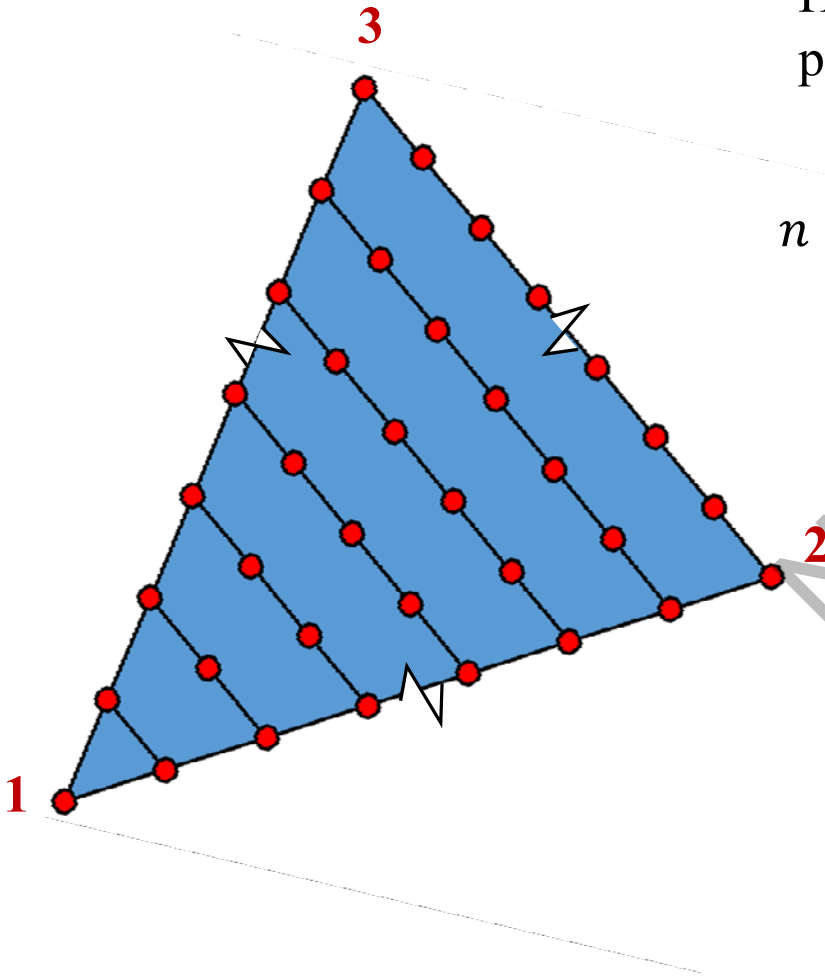


4 nodes per side
 (equally spaced) - cubic
 Total number of nodes
 $4 + 3 + 2 + 1 = 10$

Area Coordinates

Higher order triangle with k nodes (equally spaced) per sides

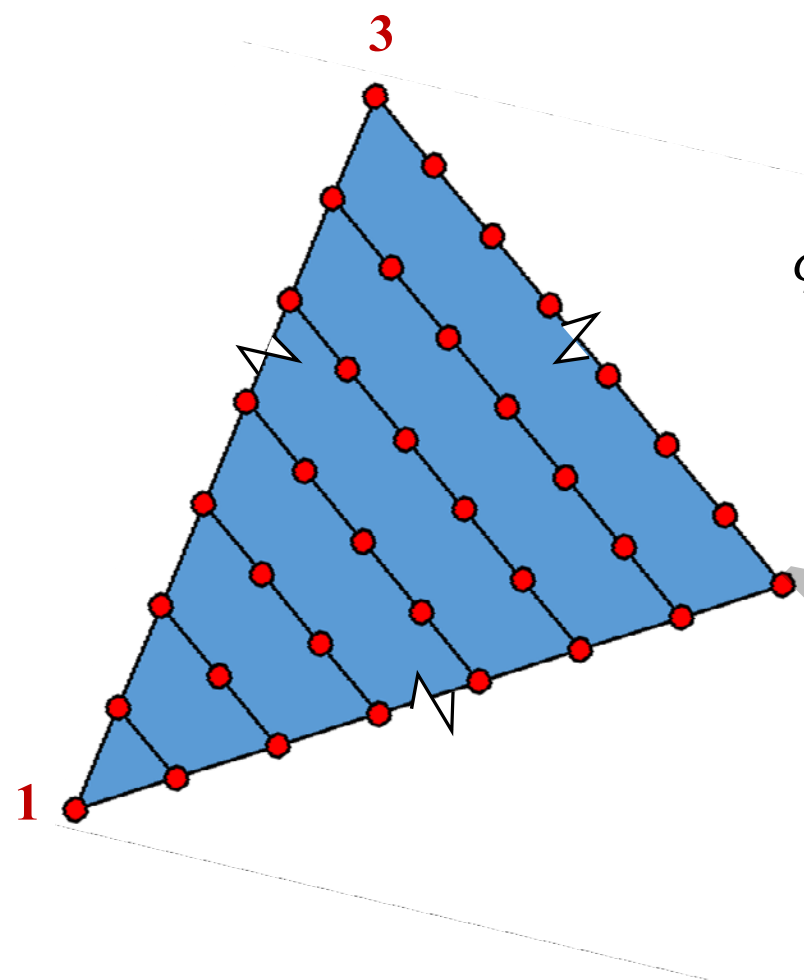
$$n = \sum_{i=0}^{k-1} (k - i) = k + (k - 1) + \cdots + 1 = \frac{1}{2}k(k + 1)$$



Area Coordinates

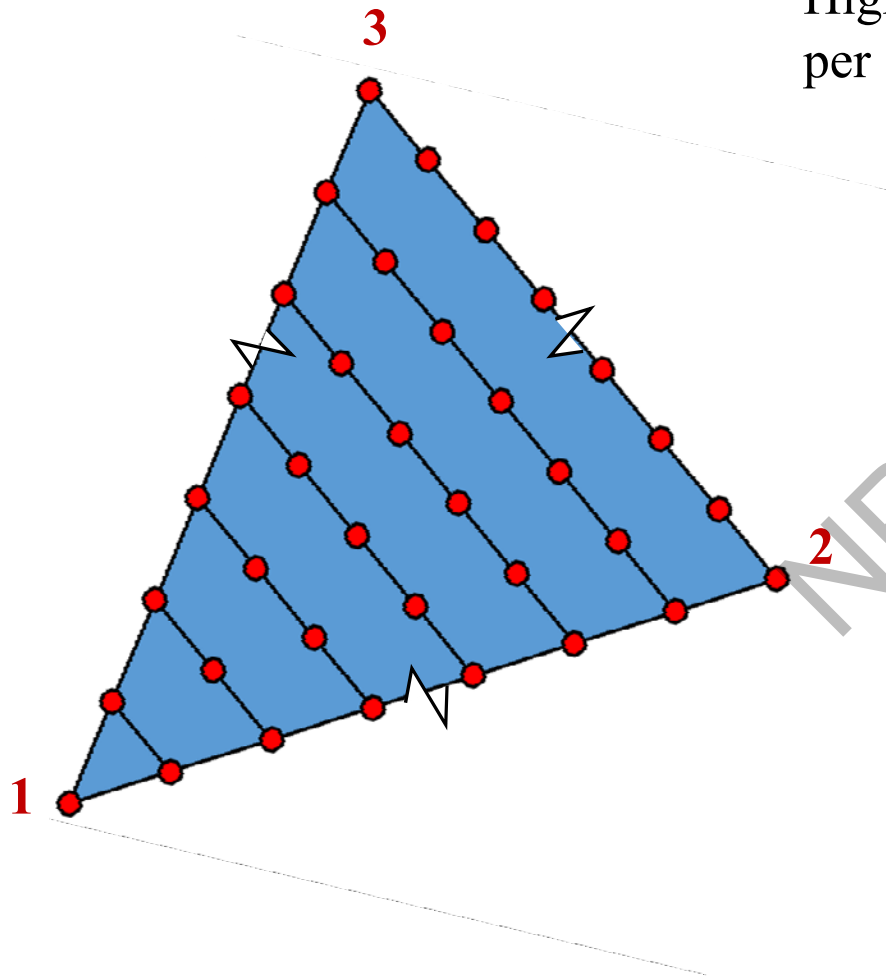
Higher order triangle with k nodes (equally spaced) per sides

$$\phi_1 = \frac{(L_1 - s_0)(L_1 - s_1)(L_1 - s_2) \cdots (L_1 - s_{k-2})}{(s_{k-1} - s_0)(s_{k-1} - s_1) \cdots (s_{k-1} - s_{k-2})}$$



Area Coordinates

Higher order triangle with k nodes (equally spaced) per sides



$$\phi_i = \prod_{j=1}^{k-1} \frac{f_j}{f_j^i}$$

f_j are the functions of L_1 , L_2 and L_3 .
 f_j^i is the value of f_j at node i



**THANK
YOU !**



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Finite Element Method

Week 10: Two-dimensional problems in linear elasticity

Lecture 49: Triangular Elements

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Recall: Interpolation function through area Coordinates

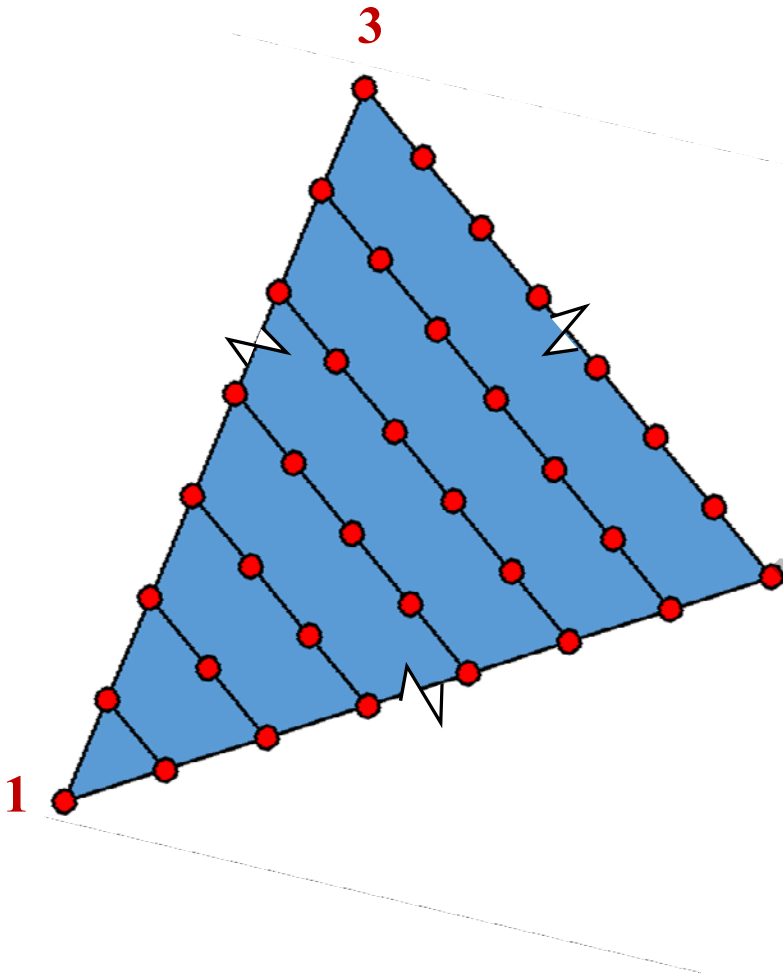
Higher order triangle with k nodes (equally spaced) per sides

$$\phi_i = \prod_{j=1}^{k-1} \frac{f_j}{f_j^i}$$

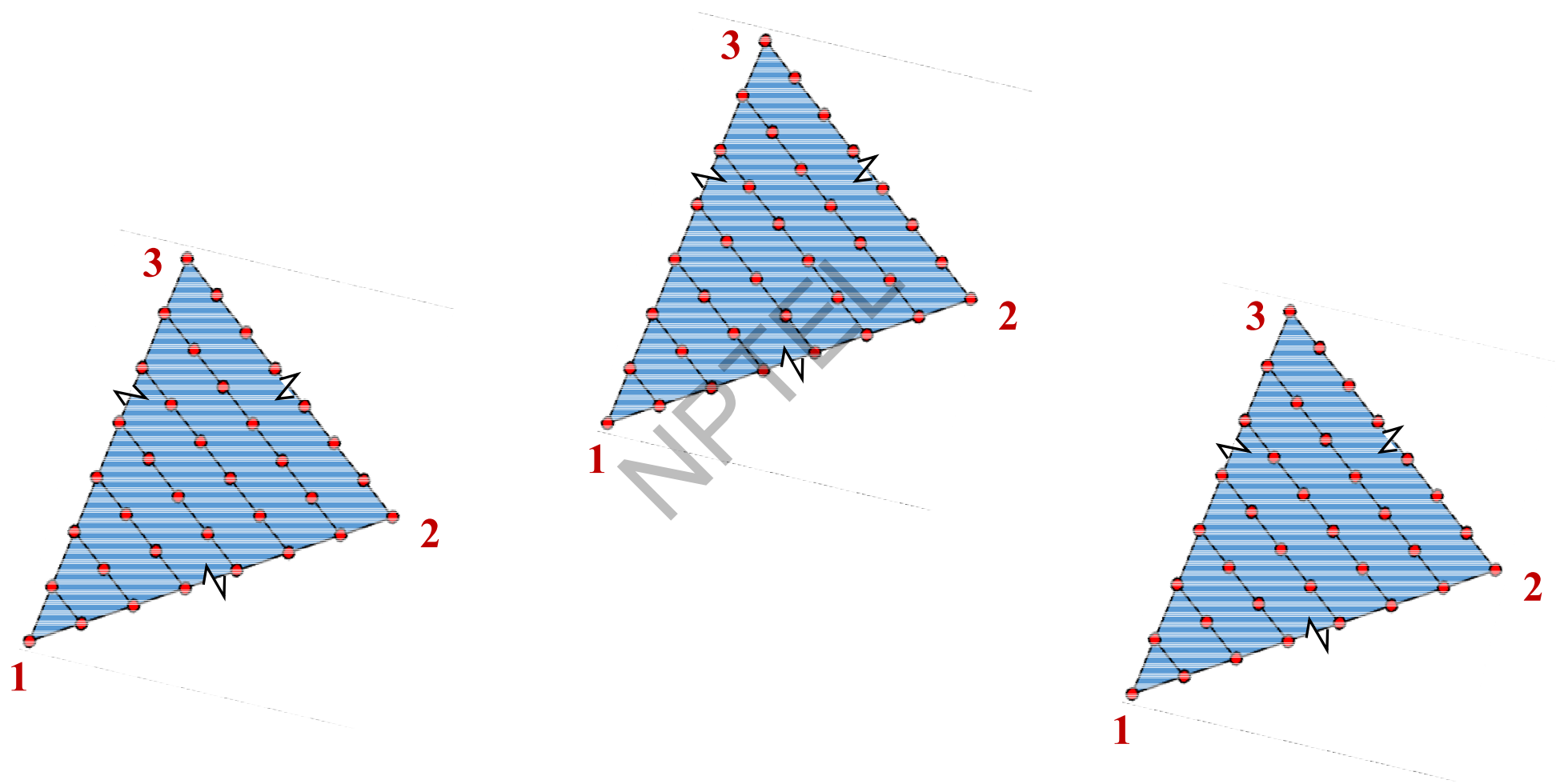
f_j are the functions of L_1, L_2 and L_3 .

f_j^i is the value of f_j at node i

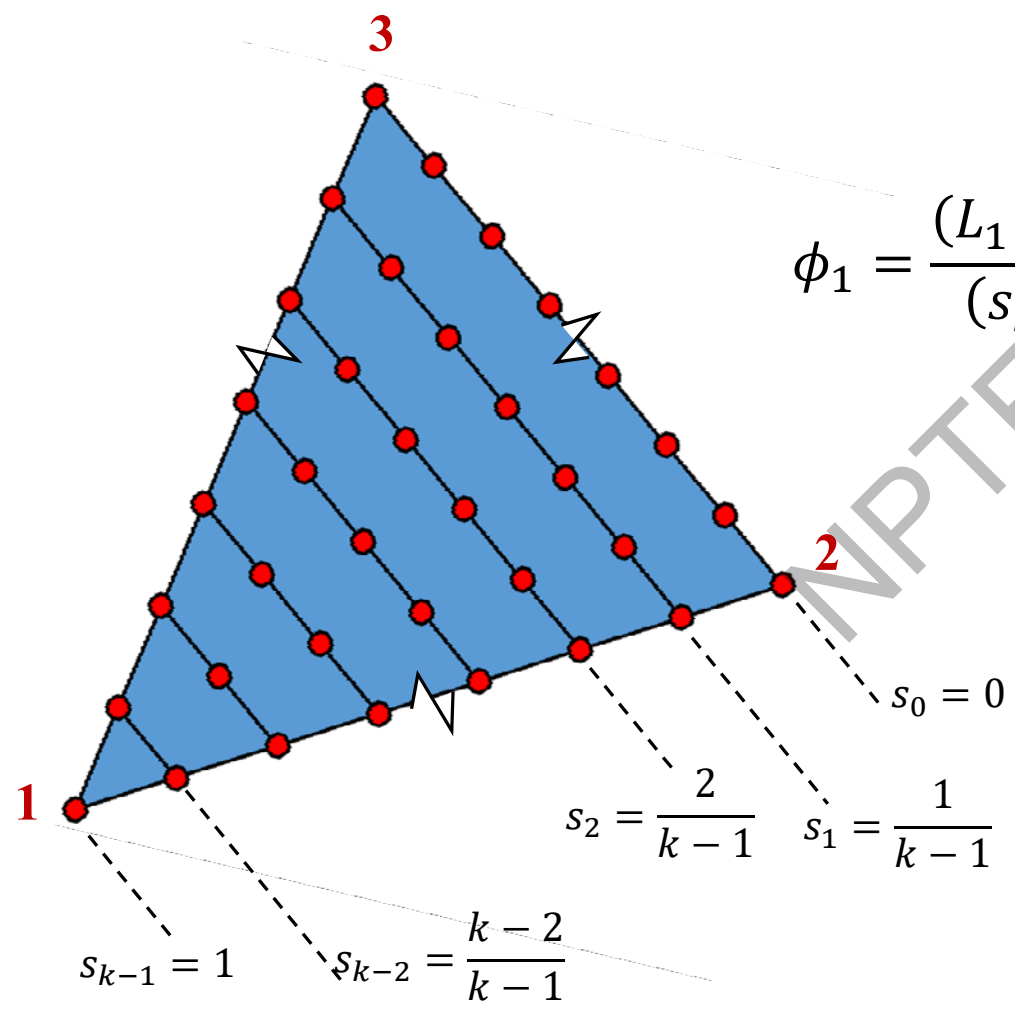
L_1, L_2 and L_3 are called area coordinates



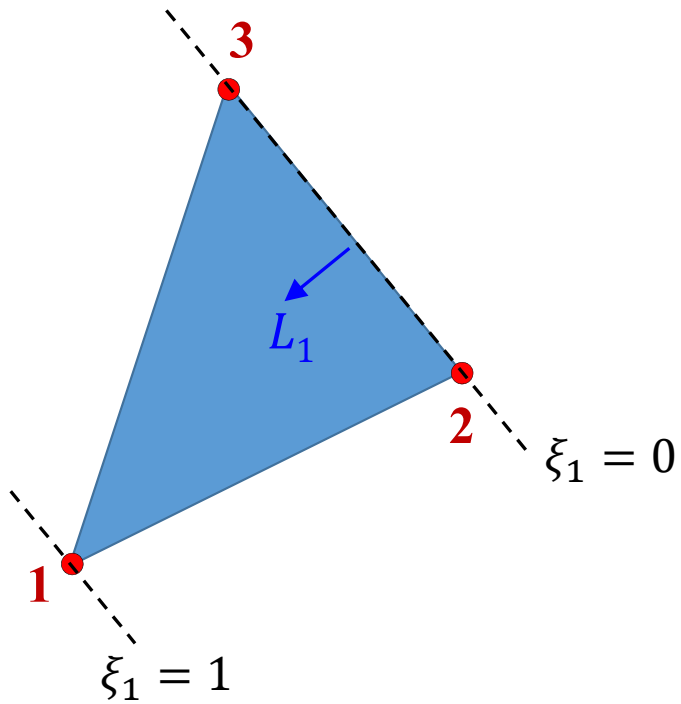
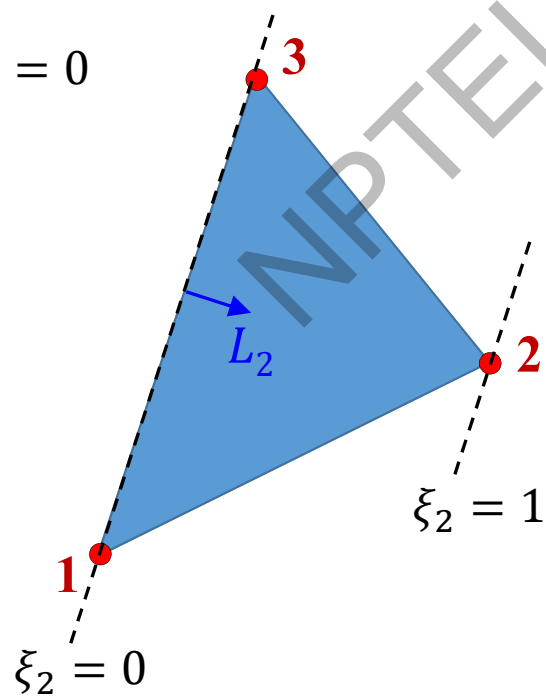
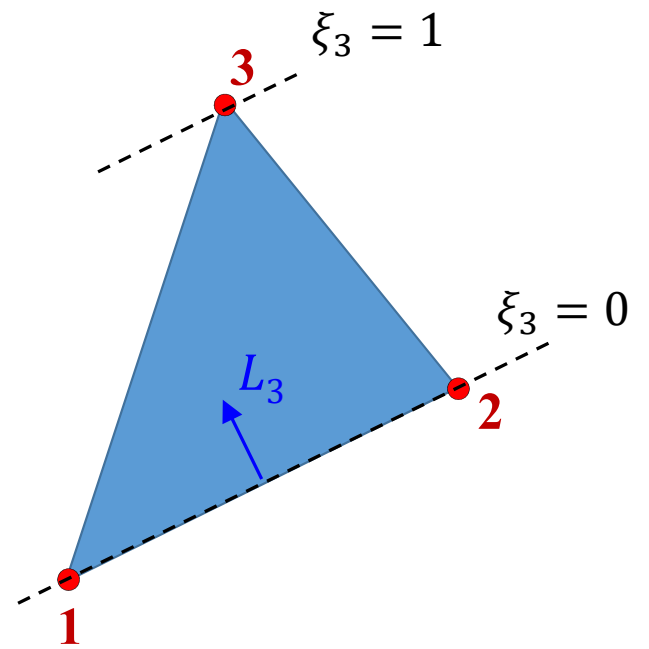
Recall: Interpolation function through area Coordinates

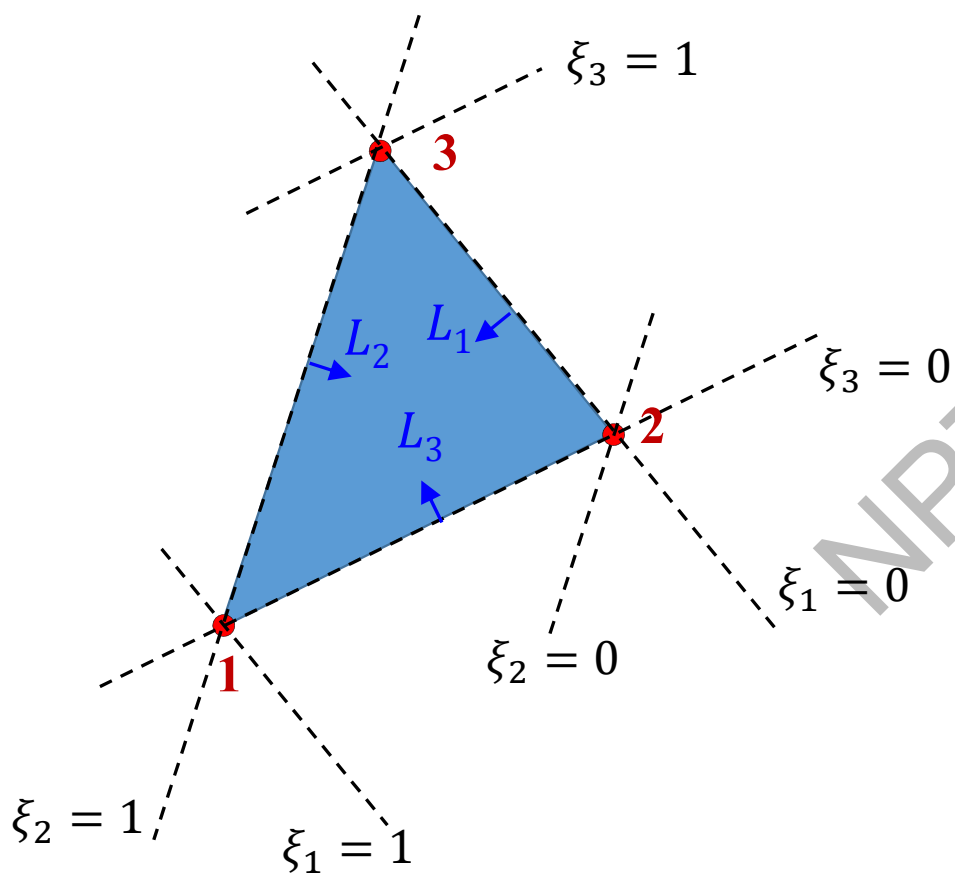
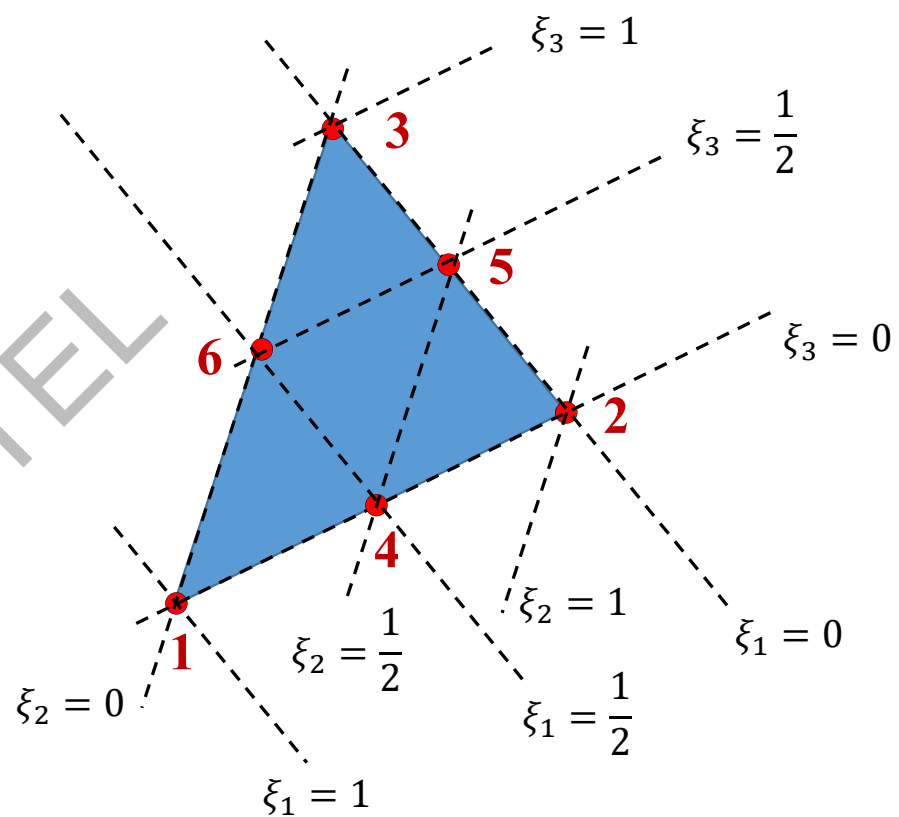


Recall: Interpolation function through area Coordinates

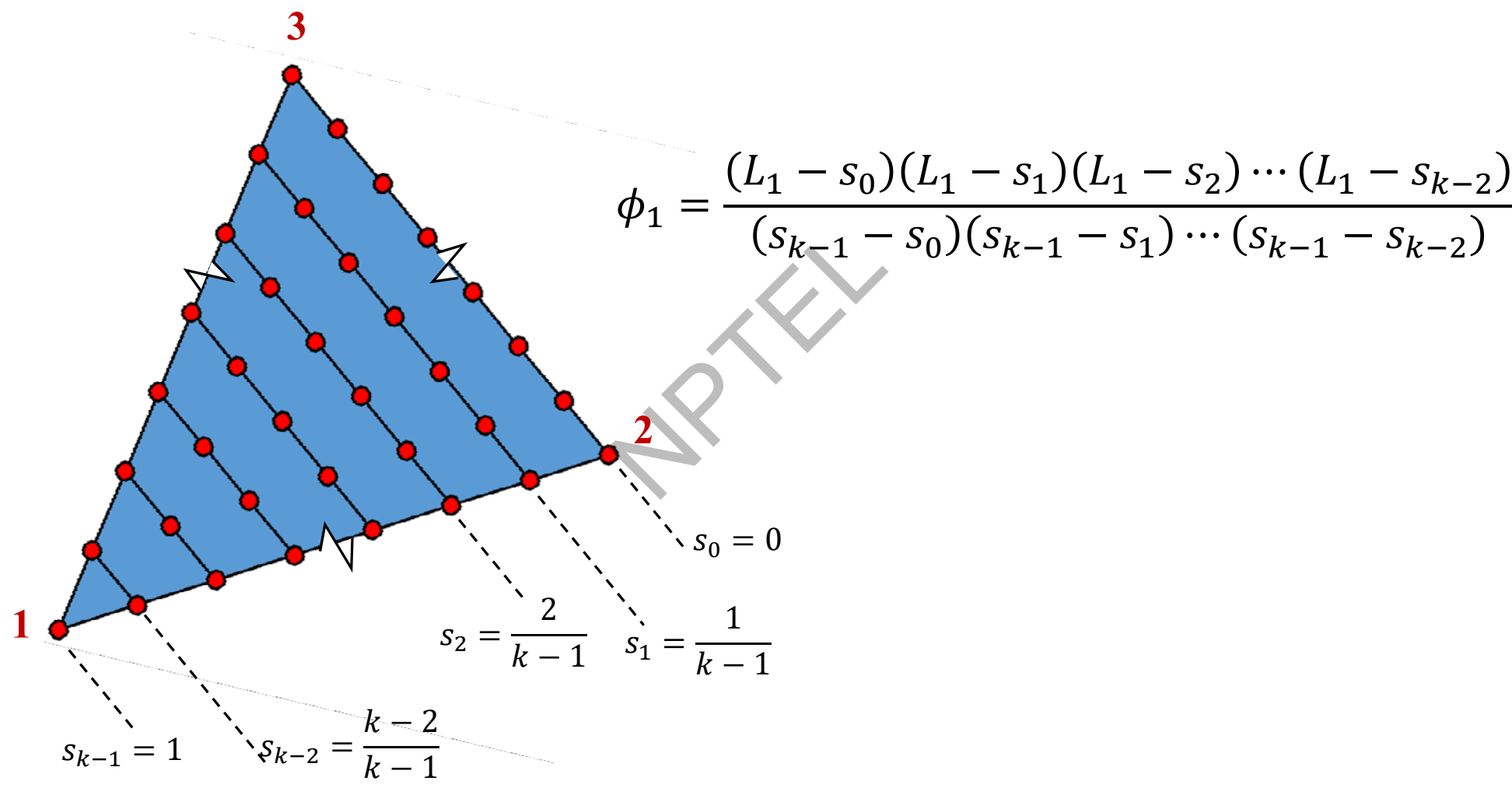


$$\phi_1 = \frac{(L_1 - s_0)(L_1 - s_1)(L_1 - s_2) \cdots (L_1 - s_{k-2})}{(s_{k-1} - s_0)(s_{k-1} - s_1) \cdots (s_{k-1} - s_{k-2})}$$

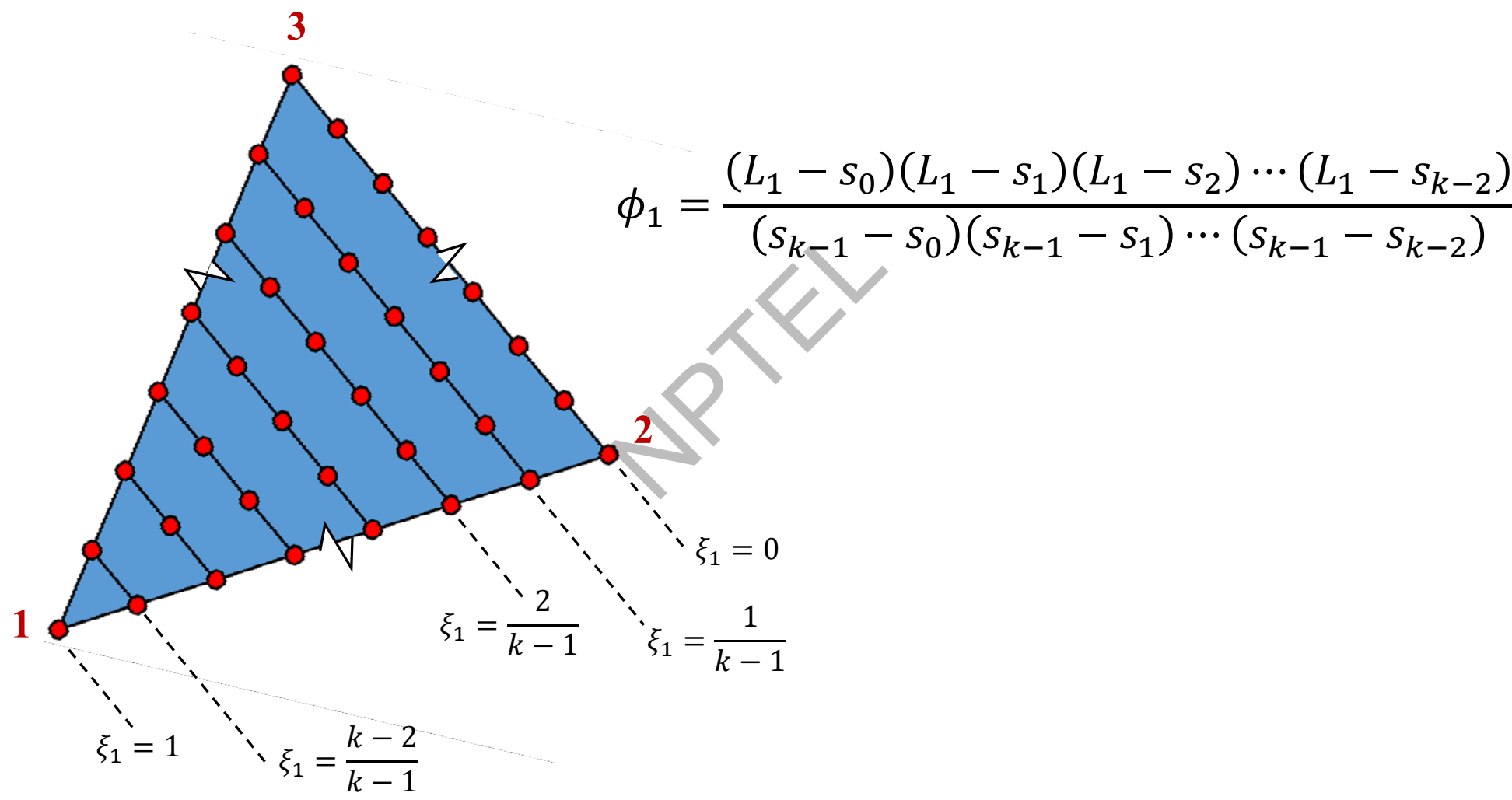




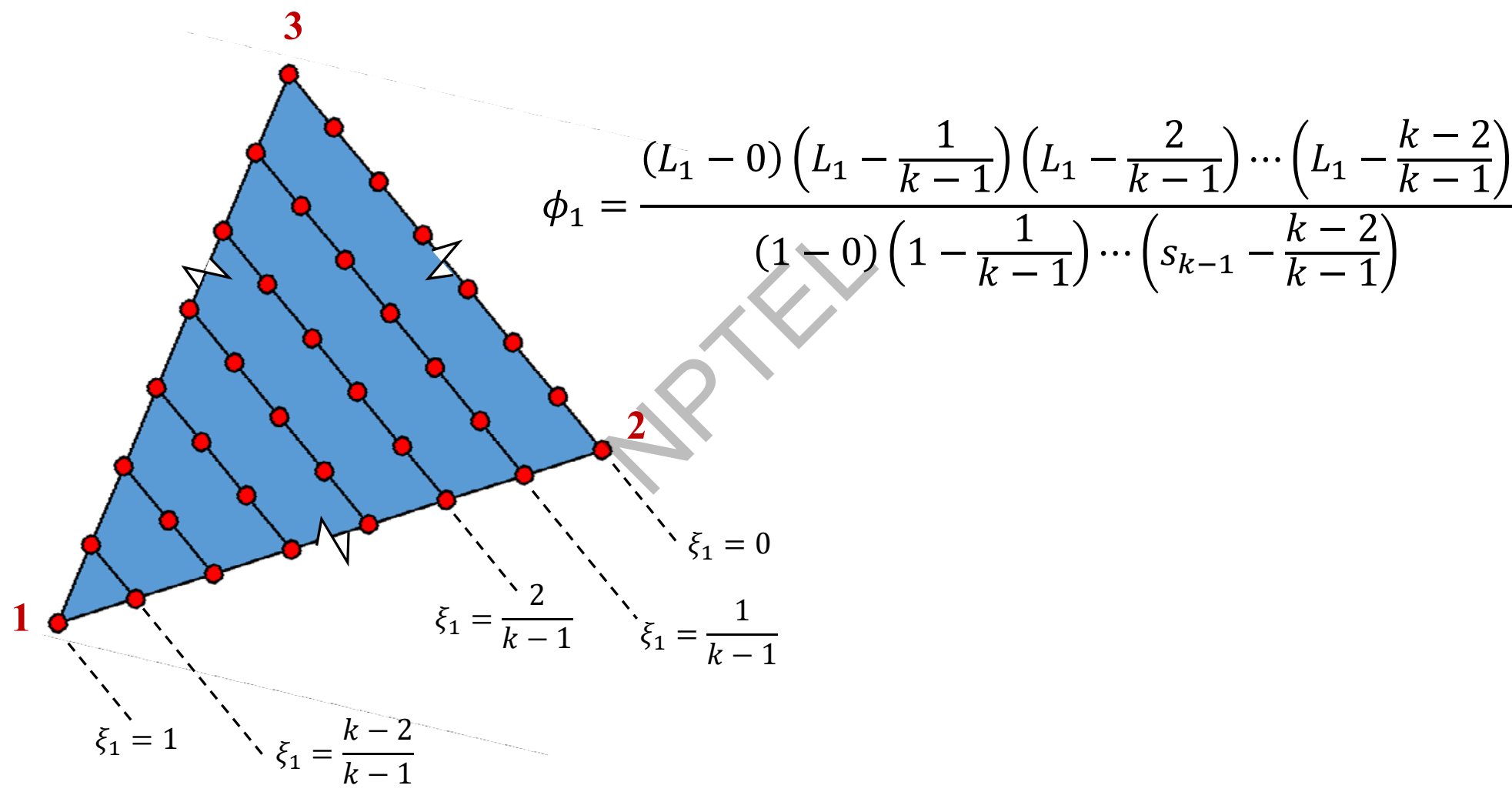
Interpolation function through area Coordinates



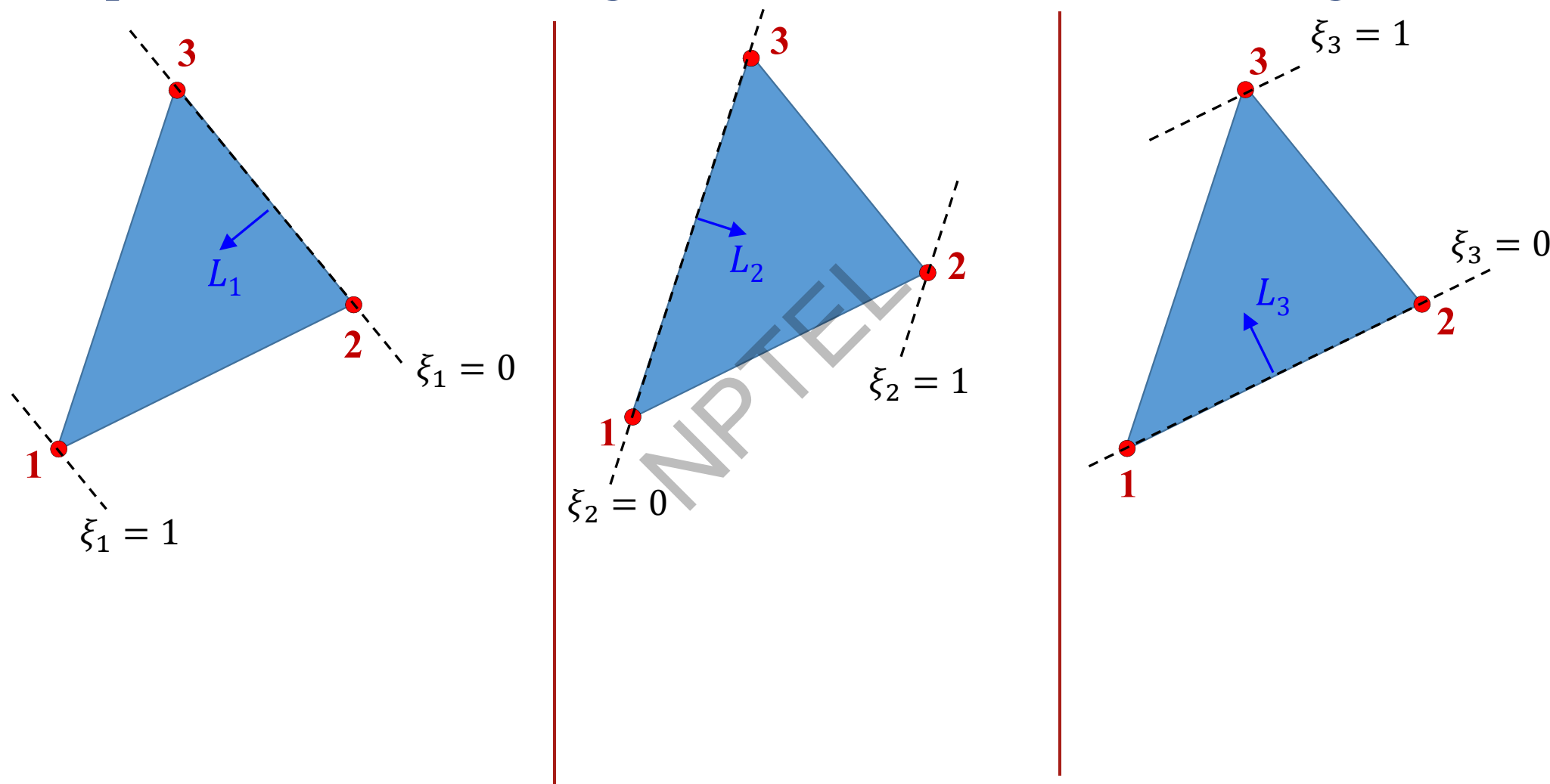
Interpolation function through area Coordinates



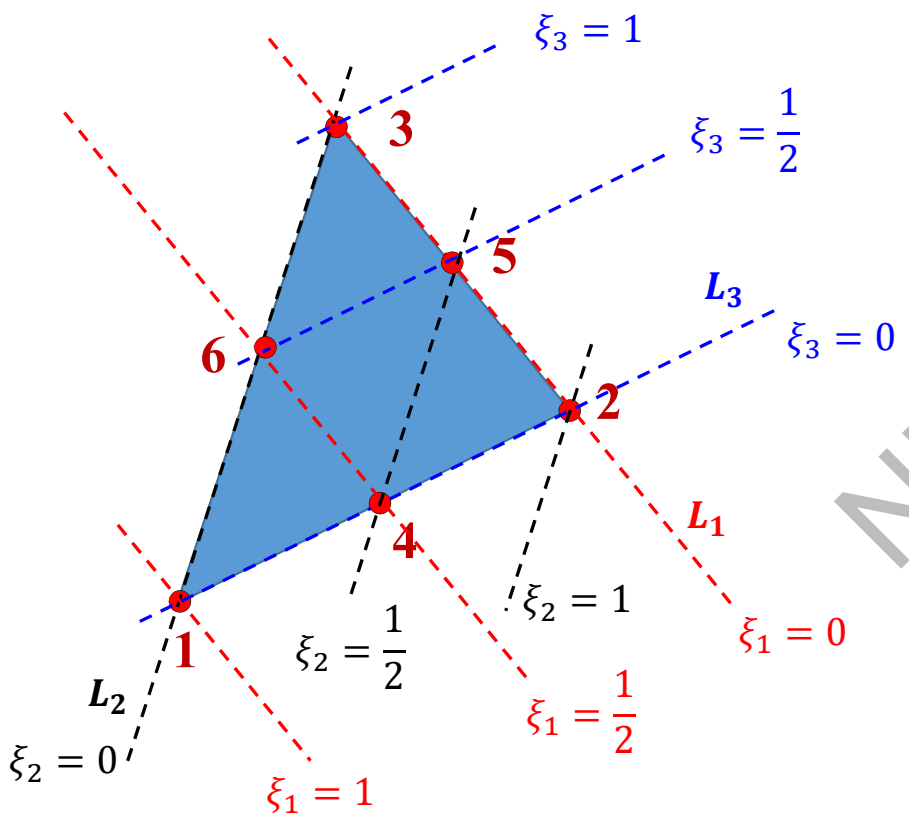
Interpolation function through area Coordinates



Interpolation function through area Coordinates: 3-node triangle

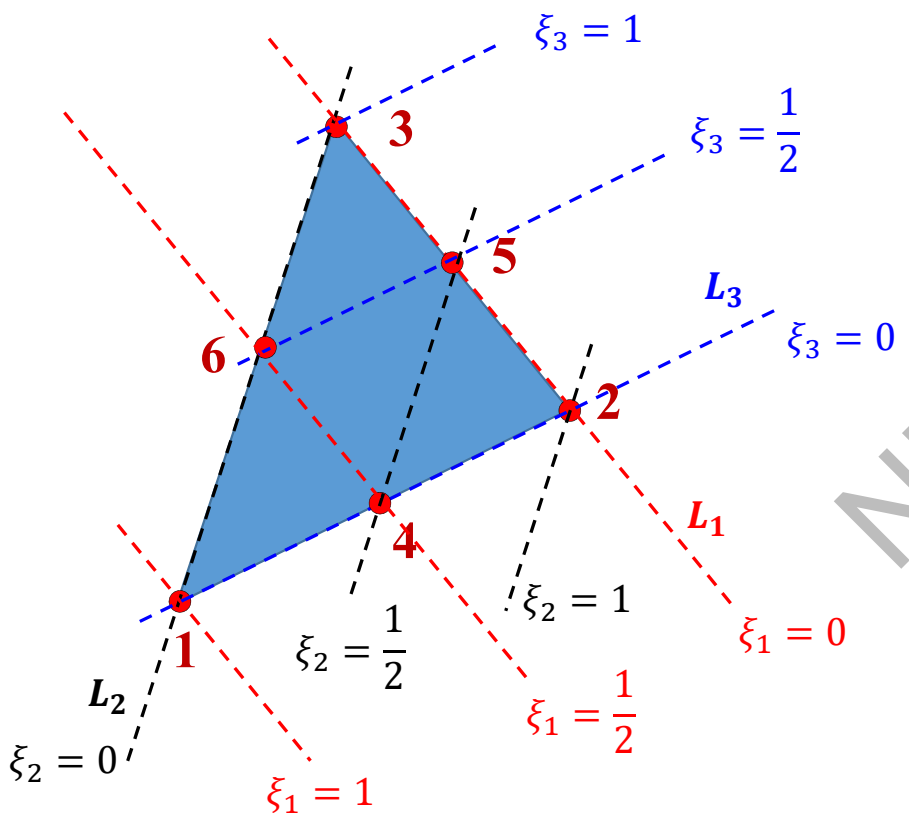


Interpolation function through area Coordinates: 6-node triangle



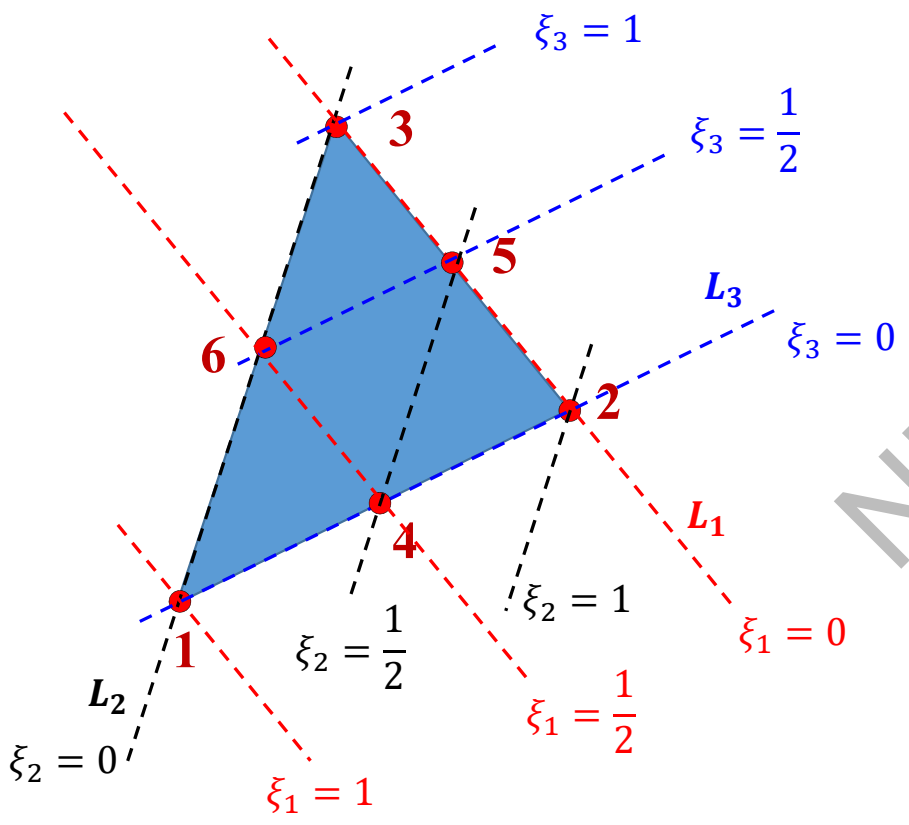
NPTTEL

Interpolation function through area Coordinates: 6-node triangle



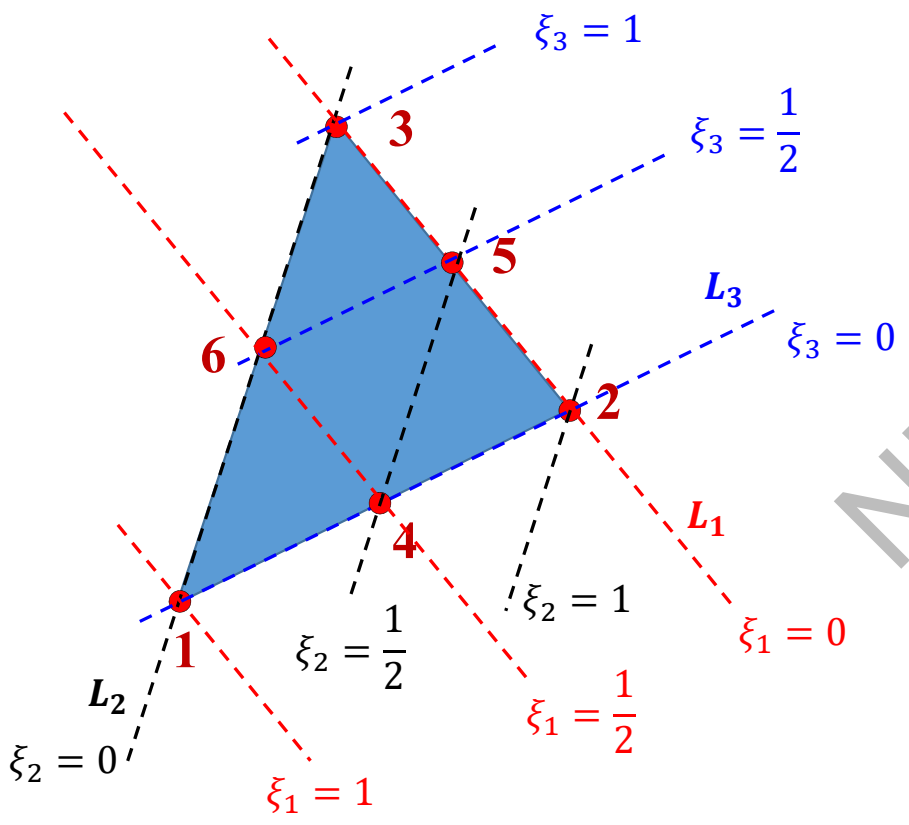
NPTTEL

Interpolation function through area Coordinates: 6-node triangle



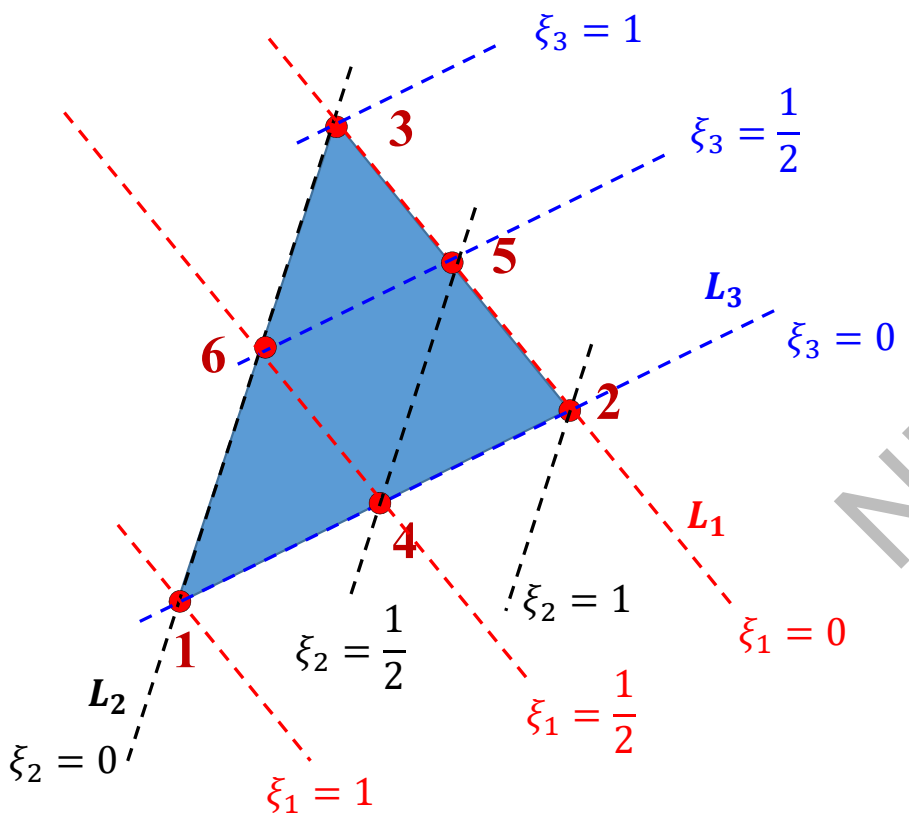
NPTTEL

Interpolation function through area Coordinates: 6-node triangle



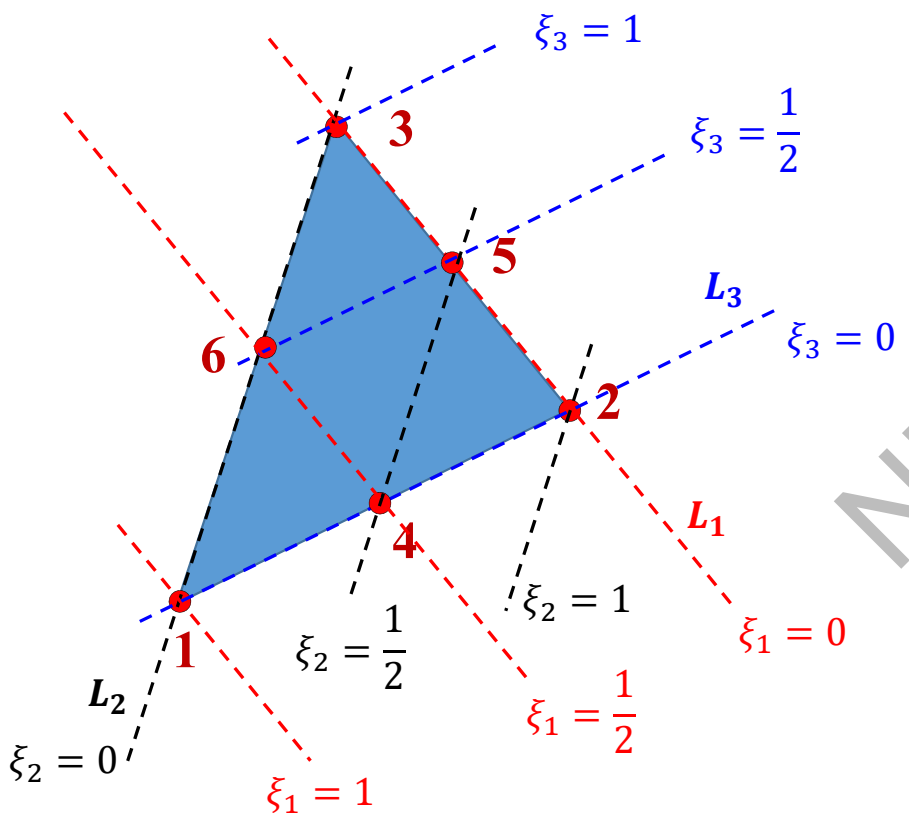
NPTTEL

Interpolation function through area Coordinates: 6-node triangle



NPTTEL

Interpolation function through area Coordinates: 6-node triangle



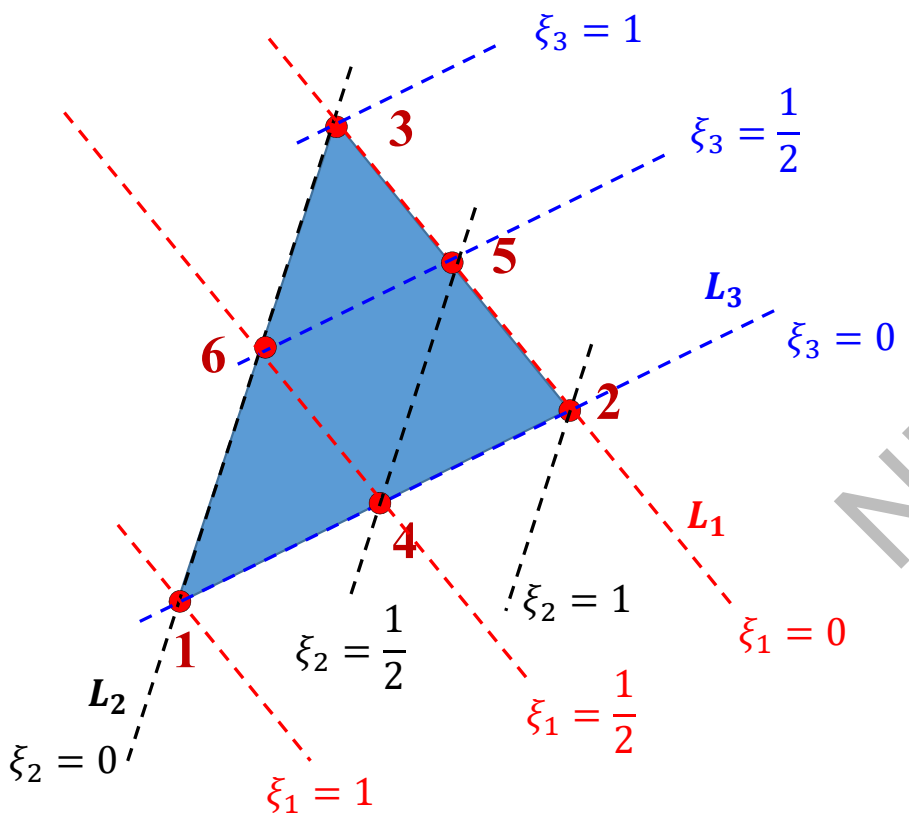
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Recall: Integration Over Triangular Domain

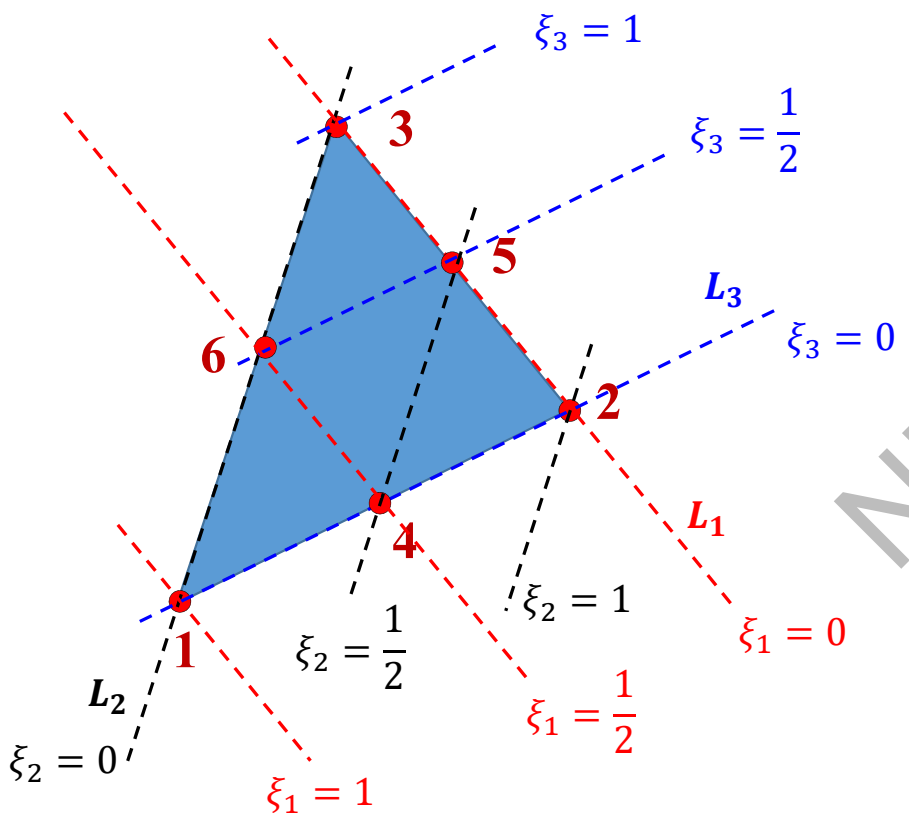
$$\int_{\Omega_e} L_1^i L_2^j L_3^k d\Omega_e = \frac{i! j! k!}{(i + j + k + 2)!} 2A^e$$

$$\int_a^b L_1^i L_2^j ds = \frac{i! j!}{(i + j + 1)!} (b - a)$$

Interpolation function through area Coordinates: 6-node triangle

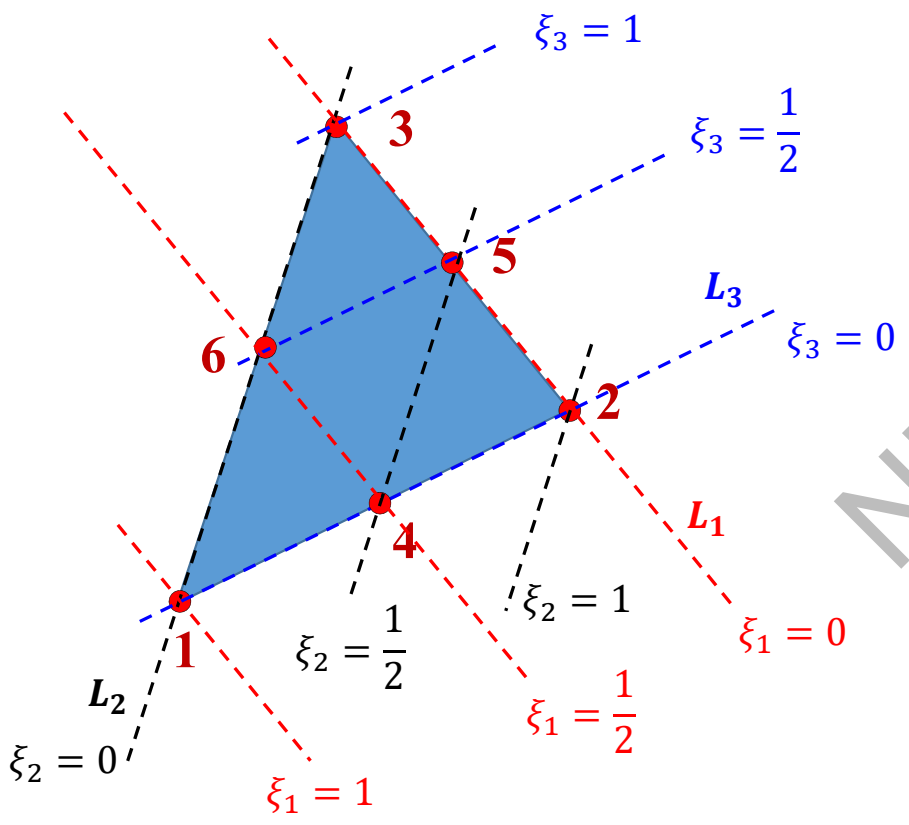


Interpolation function through area Coordinates: 6-node triangle



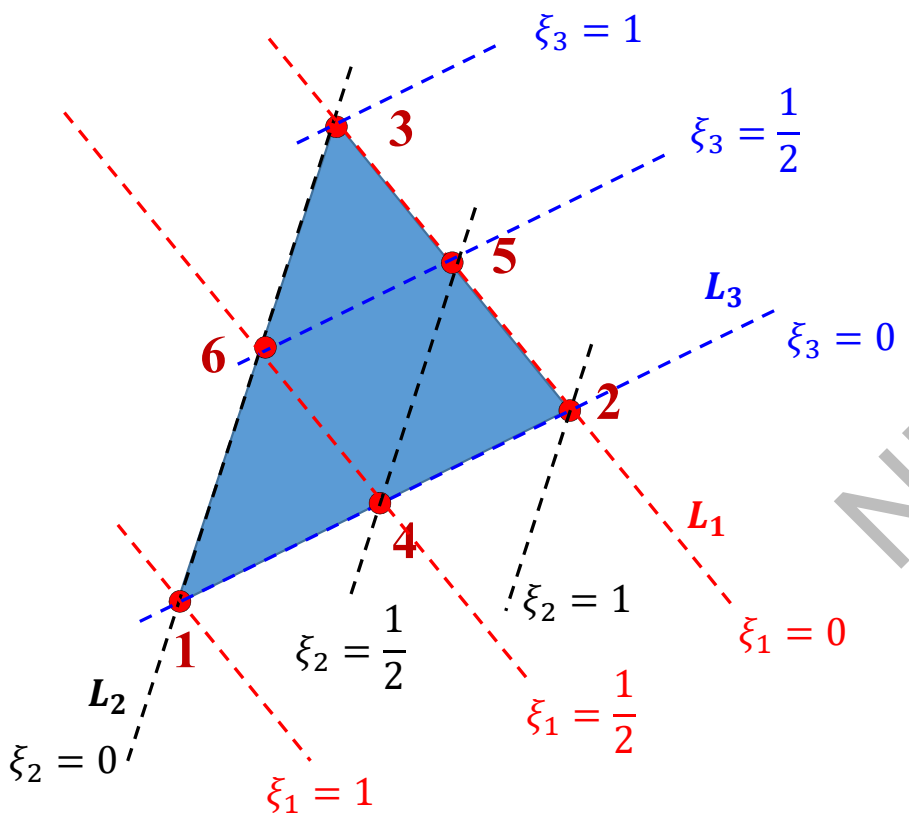
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Interpolation function through area Coordinates: 6-node triangle



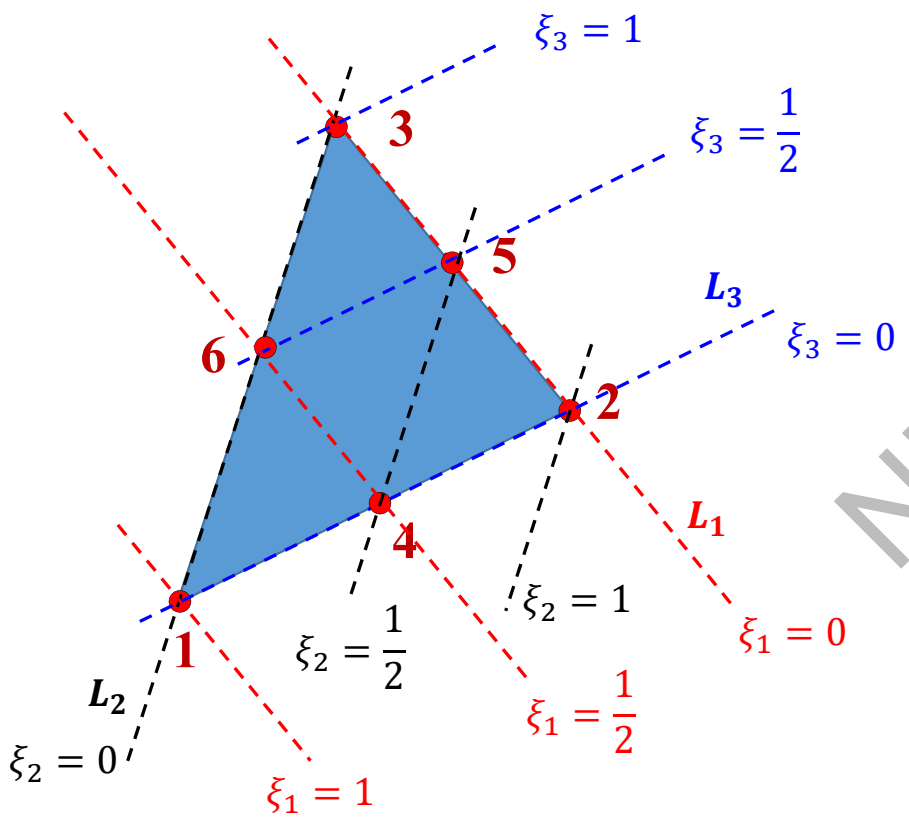
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Interpolation function through area Coordinates: 6-node triangle



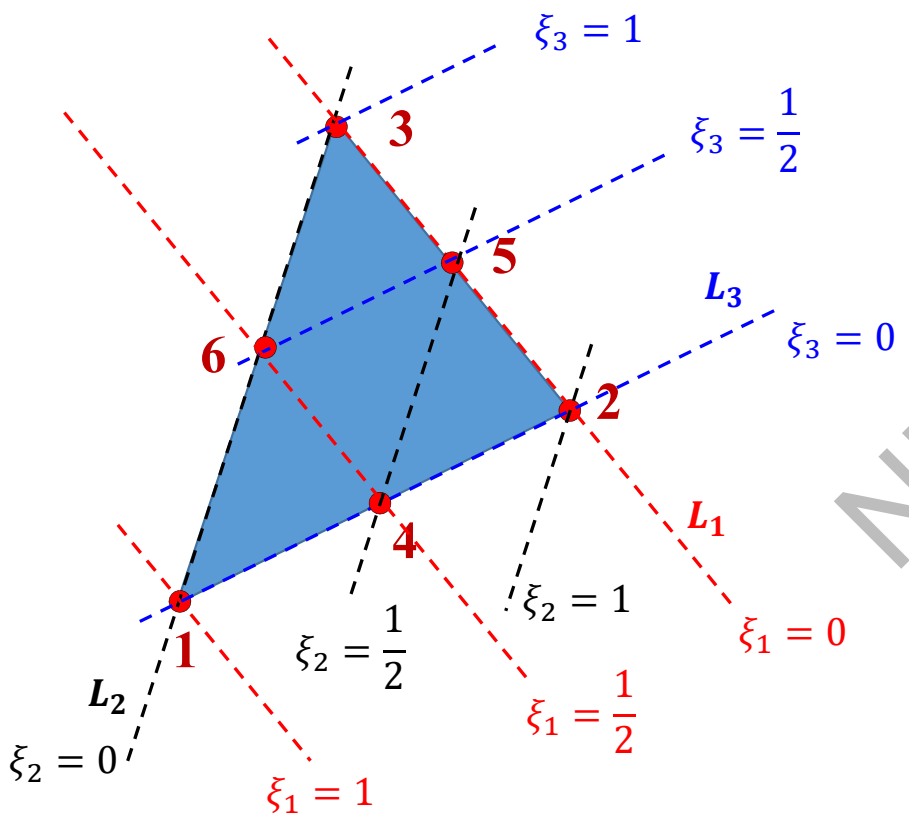
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Interpolation function through area Coordinates: 6-node triangle



NPTTEL

Interpolation function through area Coordinates: 6-node triangle



NPTTEL



**THANK
YOU !**



NPTEL ONLINE CERTIFICATION COURSES

Finite Element Method

Week 10: Two-dimensional problems in linear elasticity

Lecture 50: Examples and computer implementation

Biswanath Banerjee
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IIT Kharagpur

Recall: Plane Elasticity and Element Equation

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\}$$

$$[K^e] = \int_{\Omega^e} t_e \mathbf{B}^{eT} \mathbf{C} \mathbf{B}^e dx dy$$

$$\{f^e\} = \int_{\Omega^e} t_e \begin{bmatrix} \phi_1^e & 0 \\ 0 & \phi_1^e \\ \phi_2^e & 0 \\ 0 & \phi_2^e \\ \vdots & \vdots \\ \phi_n^e & 0 \\ 0 & \phi_n^e \end{bmatrix} \begin{bmatrix} f_x^e \\ f_y^e \end{bmatrix} dx dy$$

$$\mathbf{B}^e(x, y) = \begin{bmatrix} \frac{\partial \phi_1^e}{\partial x} & 0 & \frac{\partial \phi_2^e}{\partial x} & 0 & \dots & \frac{\partial \phi_n^e}{\partial x} & 0 \\ 0 & \frac{\partial \phi_1^e}{\partial y} & 0 & \frac{\partial \phi_2^e}{\partial y} & \dots & 0 & \frac{\partial \phi_n^e}{\partial y} \\ \frac{\partial \phi_1^e}{\partial y} & \frac{\partial \phi_1^e}{\partial x} & \frac{\partial \phi_2^e}{\partial y} & \frac{\partial \phi_2^e}{\partial x} & \dots & \frac{\partial \phi_n^e}{\partial y} & \frac{\partial \phi_n^e}{\partial x} \end{bmatrix}$$

Plane Stress

$$\mathbf{C} = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1 - \nu)/2 \end{bmatrix}$$

Plane Strain

$$\mathbf{C} = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 - \nu & \nu & 0 \\ \nu & 1 - \nu & 0 \\ 0 & 0 & (1 - 2\nu)/2 \end{bmatrix}$$

$$\{Q^e\} = \int_{\Gamma^e} t_e \begin{bmatrix} \phi_1^e & 0 \\ 0 & \phi_1^e \\ \phi_2^e & 0 \\ 0 & \phi_2^e \\ \vdots & \vdots \\ \phi_n^e & 0 \\ 0 & \phi_n^e \end{bmatrix} \begin{bmatrix} T_x^e \\ T_y^e \end{bmatrix} ds$$

Recall: Plane Elasticity and Element Equation

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\} \qquad [K^e] = \int_{\Omega^e} t_e \mathbf{B}^{eT} \mathbf{C} \mathbf{B}^e dx dy$$

$$\{f^e\} = \frac{A_e t_e}{3} \begin{Bmatrix} f_x^e \\ f_y^e \\ f_x^e \\ f_y^e \\ f_x^e \\ f_y^e \end{Bmatrix}$$

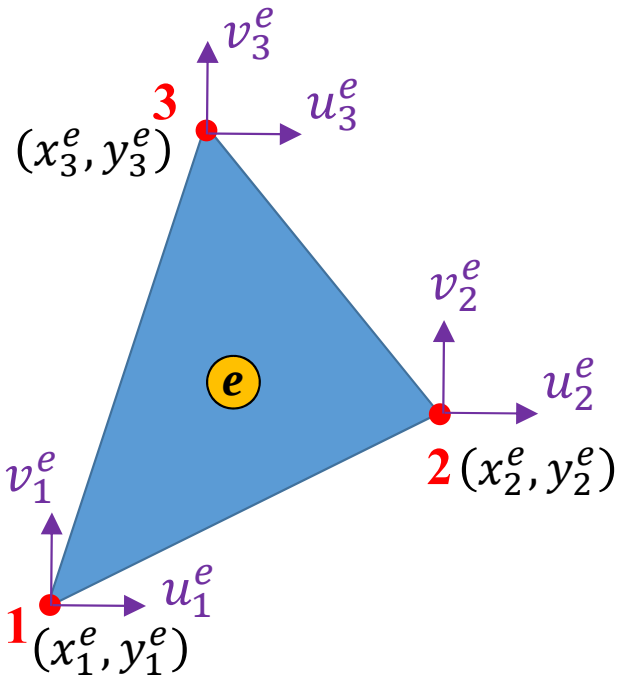
$$\{Q^e\} = \int_{\Gamma^e} t_e \begin{bmatrix} \phi_1^e & 0 \\ 0 & \phi_1^e \\ \phi_2^e & 0 \\ 0 & \phi_2^e \\ \vdots & \vdots \\ \phi_n^e & 0 \\ 0 & \phi_n^e \end{bmatrix} \begin{bmatrix} T_x^e \\ T_y^e \end{bmatrix} ds$$

$$\mathbf{B}^e(x,y) = \frac{1}{2A^e} \begin{bmatrix} y_{23}^e & 0 & y_{31}^e & 0 & y_{12}^e & 0 \\ 0 & x_{32}^e & 0 & x_{13}^e & 0 & x_{21}^e \\ x_{32}^e & y_{23}^e & x_{13}^e & y_{31}^e & x_{21}^e & y_{12}^e \end{bmatrix}$$

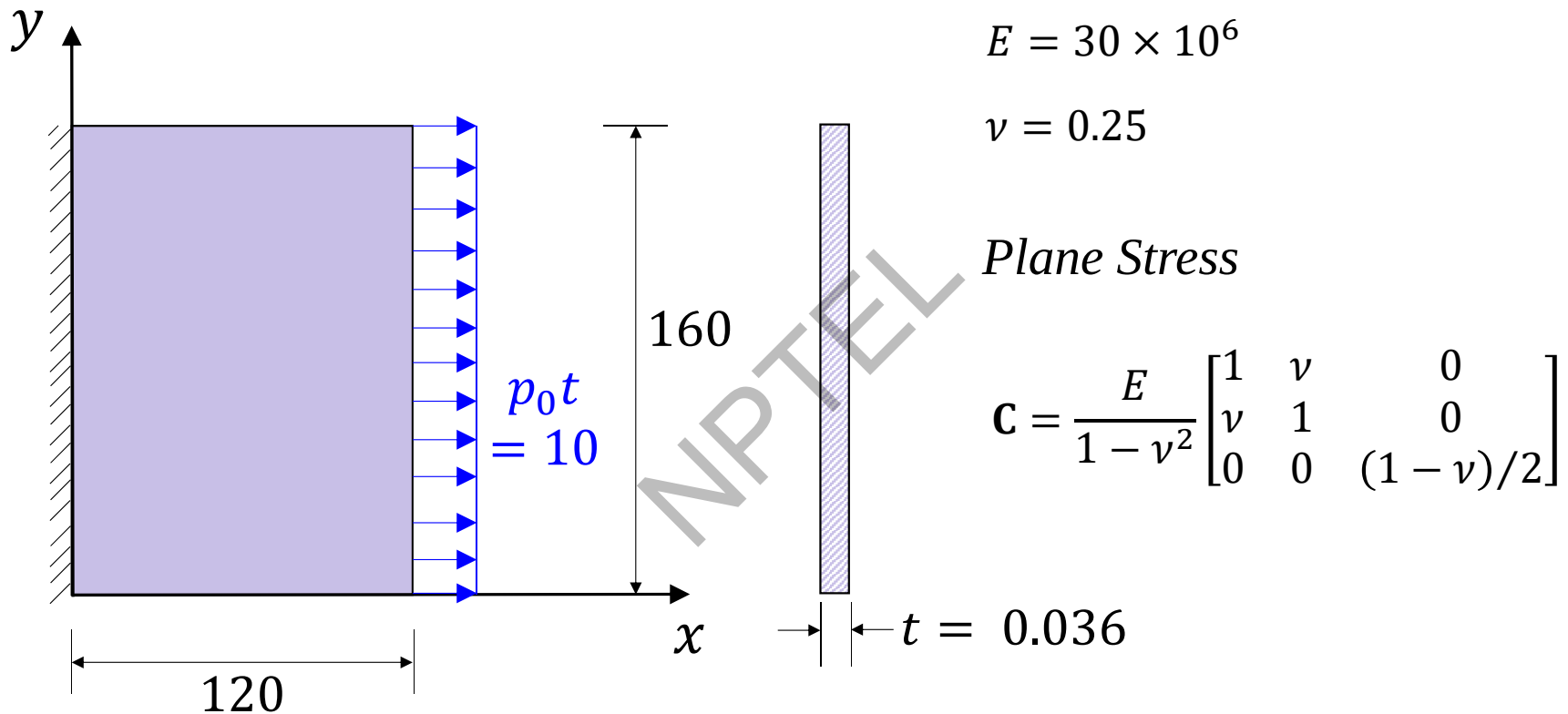
$$y_{ij}^e = y_i^e - y_j^e$$

$$x_{ij}^e = x_i^e - x_j^e$$

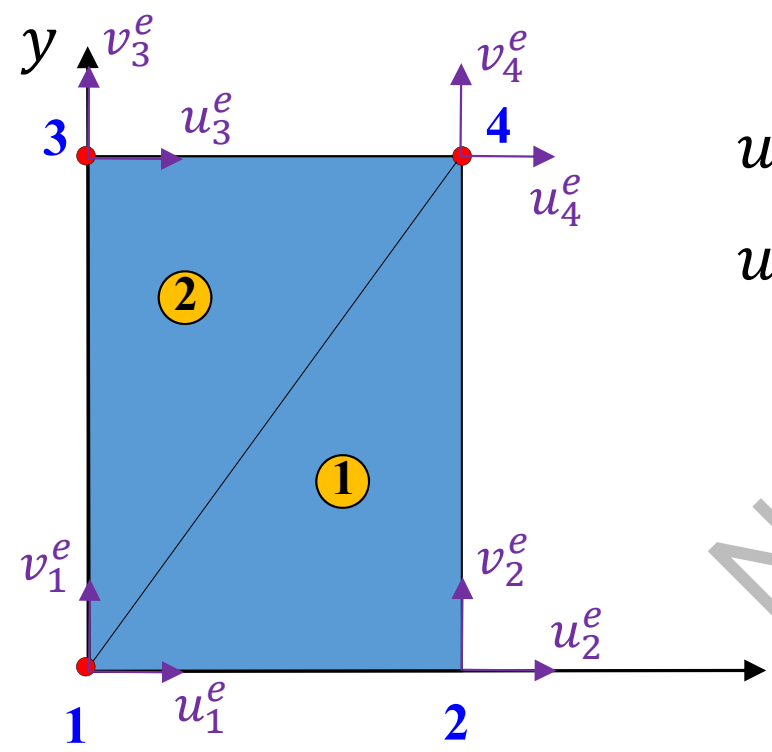
$$[K^e] = t_e A^e \mathbf{B}^{eT} \mathbf{C} \mathbf{B}^e$$



Recall: Example of Plane Elasticity Problem



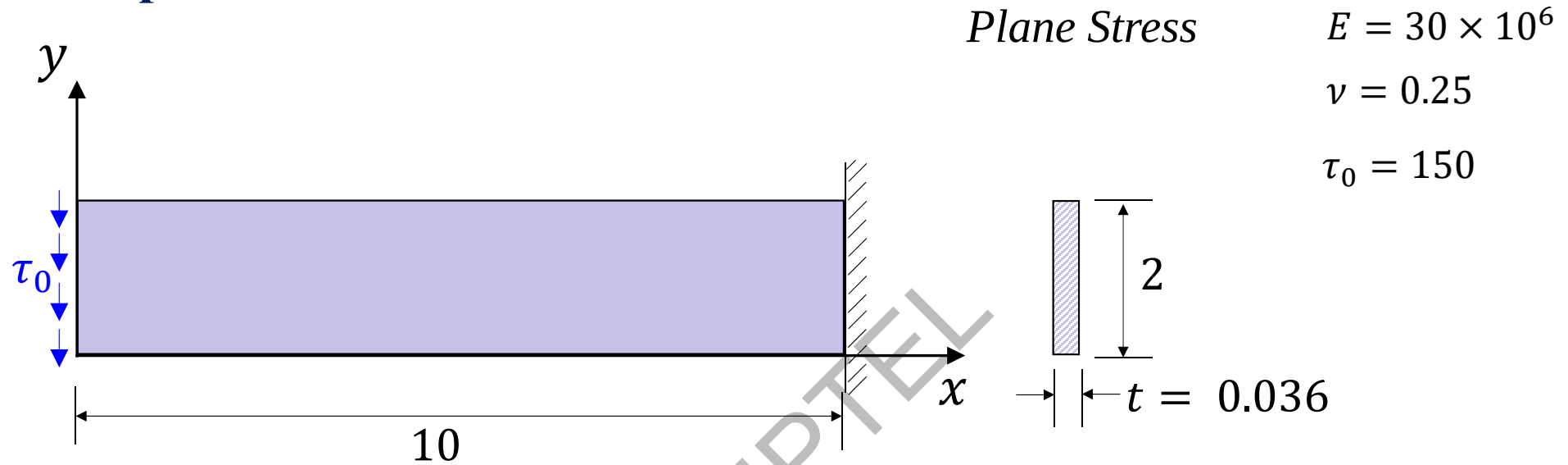
Recall: Example of Plane Elasticity Problem



$$\begin{aligned} u_1 &= v_1 = 0 \\ u_3 &= v_3 = 0 \end{aligned}$$

$$\begin{Bmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{Bmatrix} = 10^{-4} \begin{Bmatrix} 11.28 \\ 1.97 \\ 10.10 \\ -1.09 \end{Bmatrix}$$

Example 2



Elasticity solution

$$u_y(0,1) = -\frac{(\tau_0 t b)L^3}{3EI} \left\{ 1 + \frac{3(1+\nu)}{L^3} \right\}$$

$$\Rightarrow u_y(0,1) = -5.019 \times 10^{-3}$$



**THANK
YOU !**



NPTEL ONLINE CERTIFICATION COURSES

Finite Element Method

Week 10: Two-dimensional problems in linear elasticity

Lecture 51: Examples and computer implementation

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Department of Civil Engineering
IIT Kharagpur

Amit Shaw
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Recall: Plane Elasticity and Element Equation

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\}$$

$$[K^e] = \int_{\Omega^e} t_e \mathbf{B}^{eT} \mathbf{C} \mathbf{B}^e dx dy$$

$$\{f^e\} = \int_{\Omega^e} t_e \begin{bmatrix} \phi_1^e & 0 \\ 0 & \phi_1^e \\ \phi_2^e & 0 \\ 0 & \phi_2^e \\ \vdots & \vdots \\ \phi_n^e & 0 \\ 0 & \phi_n^e \end{bmatrix} \begin{bmatrix} f_x^e \\ f_y^e \end{bmatrix} dx dy$$

$$\mathbf{B}^e(x, y) = \begin{bmatrix} \frac{\partial \phi_1^e}{\partial x} & 0 & \frac{\partial \phi_2^e}{\partial x} & 0 & \dots & \frac{\partial \phi_n^e}{\partial x} & 0 \\ 0 & \frac{\partial \phi_1^e}{\partial y} & 0 & \frac{\partial \phi_2^e}{\partial y} & \dots & 0 & \frac{\partial \phi_n^e}{\partial y} \\ \frac{\partial \phi_1^e}{\partial y} & \frac{\partial \phi_1^e}{\partial x} & \frac{\partial \phi_2^e}{\partial y} & \frac{\partial \phi_2^e}{\partial x} & \dots & \frac{\partial \phi_n^e}{\partial y} & \frac{\partial \phi_n^e}{\partial x} \end{bmatrix}$$

Plane Stress

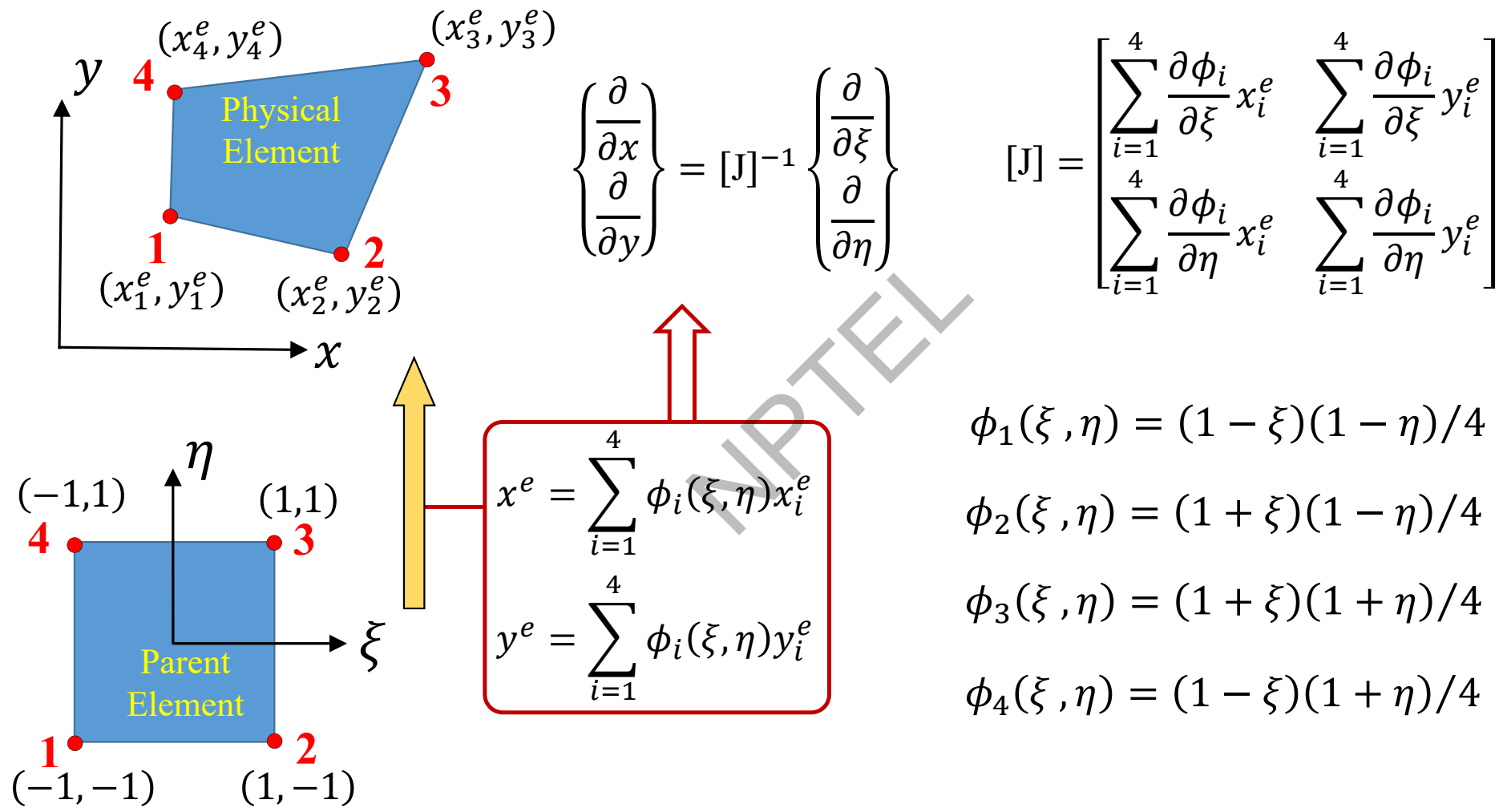
$$\mathbf{C} = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1 - \nu)/2 \end{bmatrix}$$

Plane Strain

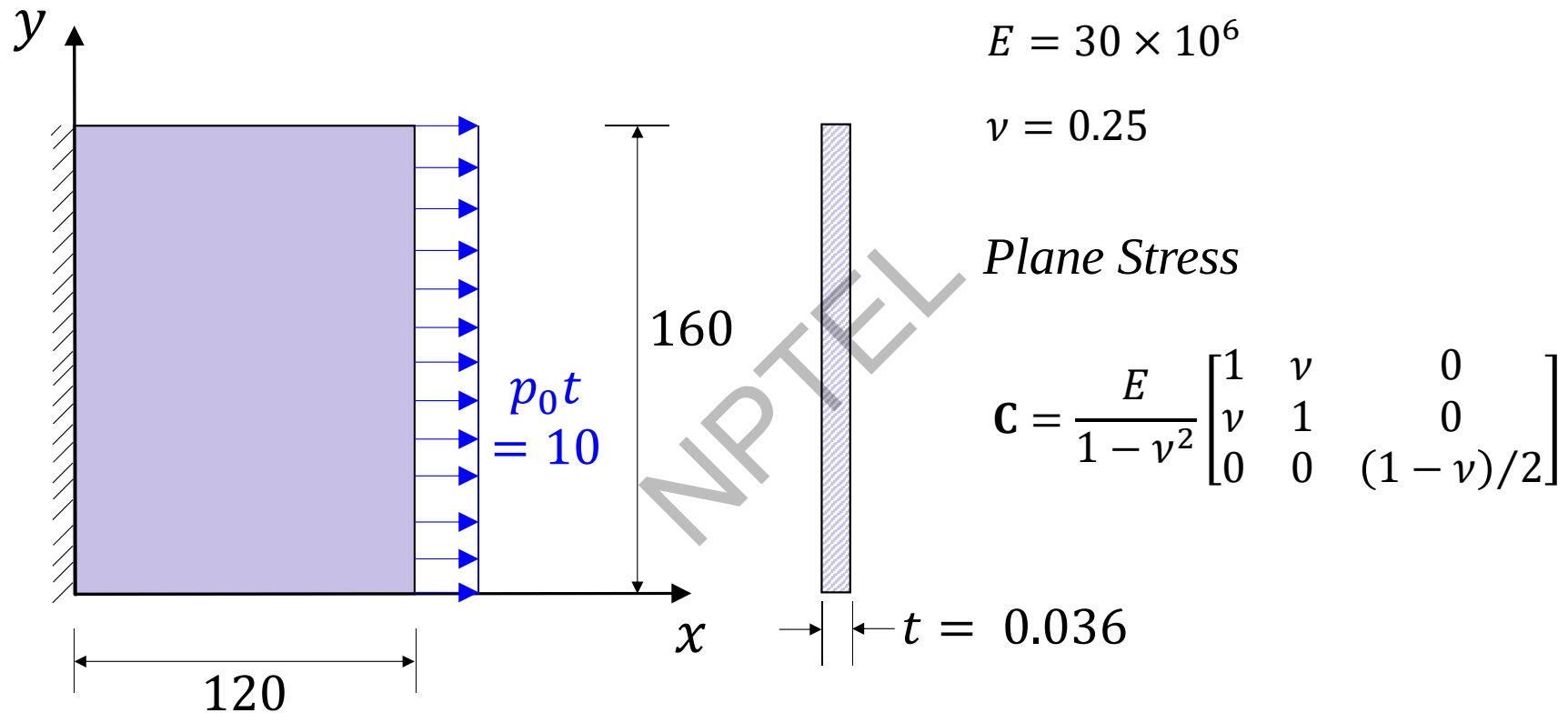
$$\mathbf{C} = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 - \nu & \nu & 0 \\ \nu & 1 - \nu & 0 \\ 0 & 0 & (1 - 2\nu)/2 \end{bmatrix}$$

$$\{Q^e\} = \int_{\Gamma^e} t_e \begin{bmatrix} \phi_1^e & 0 \\ 0 & \phi_1^e \\ \phi_2^e & 0 \\ 0 & \phi_2^e \\ \vdots & \vdots \\ \phi_n^e & 0 \\ 0 & \phi_n^e \end{bmatrix} \begin{bmatrix} T_x^e \\ T_y^e \end{bmatrix} ds$$

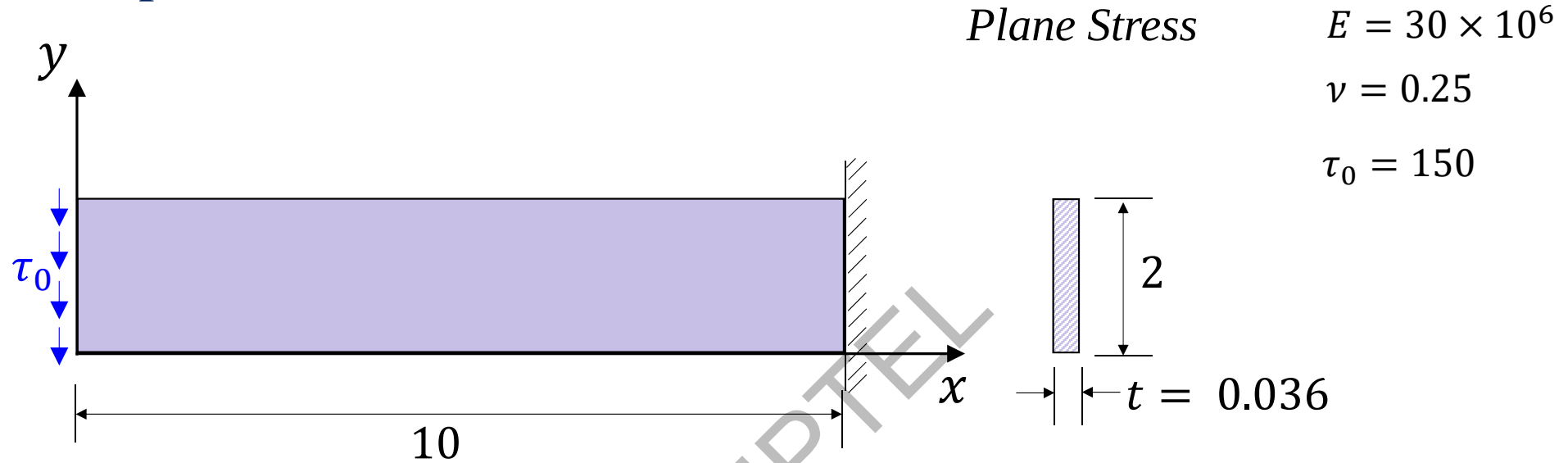
Recall: Isoparametric Formulation



Recall: Example of Plane Elasticity Problem



Example 2



Elasticity solution

$$u_y(0,1) = -\frac{(\tau_0 t b)L^3}{3EI} \left\{ 1 + \frac{3(1+\nu)}{L^3} \right\}$$

$$\Rightarrow u_y(0,1) = -5.019 \times 10^{-3}$$



**THANK
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